

Damping Effects on the Transverse Motions of Axially-loaded Beams Carrying Uniform Distributed Load

Oluwatoyin Kehinde Ogunbamike*

Department of Mathematical Sciences, Olusegun Agagu University of Science and Technology, Okitipupa, Nigeria

*Corresponding author: ogunbamike2005@gmail.com

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Abstract: In this study, the dynamic analysis of a clamped-clamped Rayleigh beam under moving distributed loads is investigated. The solution technique is based on the generalized finite integral transform and a modification of the Struble's asymptotic technique. Analytical solutions and numerical analysis showed that higher values of axial force, damping due to strain resistance and rotatory inertia reduce the response amplitudes of the beam. It is observed that the influence of structural parameters such as axial force, mass ratio and viscous damping have significant effects on the transverse motion and the critical velocity of the elastic structure carrying moving distributed load. Furthermore, it is also found that for the same natural frequency, the critical velocity for the moving distributed mass problem is smaller than that of moving distributed force problem. Hence resonance conditions for the moving distributed mass problem are reached prior to those of moving distributed force problem. Finally, the accuracy of the solutions obtained is numerically validated by comparison studies with other available cases in literature.

Keywords: Axial force; Clamped-clamped beam; Critical velocity; Resonance; Viscous damping.

1. INTRODUCTION

This paper assesses a fundamental structural problem of the vibration of axially loaded beam and traversed by distributed masses [1-4]. It is sequel to an earlier paper [5] that considers the dynamic analysis of simply supported thick beams on elastic foundation. In particular, this paper is a generalization of the theory advanced in [5]. In modern design and analysis of structures, vibration analysis of elastic structures represents an important research topic in the framework of structural and mechanical engineering as it has long been studied in order to characterize the dynamic behaviour of continuous systems. More specifically, the behaviour of elastic structures such as beams under various forms of moving loads has attracted the attention of many researchers [6-10]. A basic understanding of the complexity of the dynamic interactions between structural members and the masses traversing them is very vital as it helps in controlling the structural vibrations and saves operations of such dynamical system. Thus, a lot of literature exists in this subject when the force effect of the vibrating system is considered [11-15]. The behaviour of beam structures under moving load, in general, become complex when the inertia effect of the moving load is taken into consideration [16]. Thus, most of the studies available in literature are those in which this effect has been neglected. This is due, at least in parts, to the great amount of computational labour which is required both to set up and to solve the necessary equations. One important problem that arises when the inertia effects of the masses are considered is the singularity which occurs in the inertia terms of the governing differential equation of motion. Notable studies available in literature include the work of Sadiku and Leipholz [17], Oni [18], Gbadeyan and Oni [19], Lee and Ng [20], Savin [21] and Rao [22].

In these studies, effects of axial force were not taken into considerations; even such impacts can be significant on the frequency response of structural elements. The behaviour of such elastic structures like beams under axial load is a classical problem studied mainly in the context of static or dynamic stability. There have been some early studies of vibrating beams under axial loading [23-27], where the effect of increasing the axial loading on the mode shapes and natural frequencies of the beam was investigated. There has been some work already done for large amplitude vibration. Bhashyam and Prathap [28] used the Galerkin finite element method to study nonlinear vibration, and Özkaya [29] calculated the response of a beam mass system with clamped ends by applying a method known as the method of multiple scales. Oni and Omolofe [30] studied the dynamic response of axially prestressed Rayleigh beam on elastic foundation and subjected to concentrated masses travelling at varying velocity. Fritzkowski [31] considered the transverse in-plane vibrations of a beam which is a part of symmetrical triangular frame. He presented a mathematical model based on the Hamilton principle formulated by large deflections of the beam subjected to dynamic axial excitation. The effectiveness of Adomian Decomposition Method and Differential Transform

Method on free vibrations of axial-loaded Timoshenko beams resting on visco-elastic foundation is investigated by Bozyigit *et al.* [32]. Recently, Siqueira *et al.* [33] analyzed the free vibration of an Euler-Bernoulli beam resting on two-parameter foundation subject to axial load. The frequency equations are obtained for several boundary conditions and a finite element is developed by mean of the Rayleigh-Ritz method using a cubic approximate polynomial.

In all these aforementioned works, the effect of damping on the vibration of the dynamical system has been neglected and only numerical or semi-numerical techniques have been employed to solve the governing equation due to the rigour of the load-structure interaction and complex nature of the resulting equation. Since every system that possesses mass and elasticity is capable of oscillating; therefore, damping is present in every real oscillating system and its effects occur when energy is removed from the system. Indeed, considering the significant influence of damping on vibration characteristics of continuous beam, deep investigations on the effect of viscous damping and damping due to strain resistance of the beam material have been conducted in this study.

2. MATHEMATICAL MODEL

Consider the flexural motion of a uniform finite beam resting on an elastic foundation and carrying masses, M_i . The masses is assumed to touch the beam at time $t = 0$ and travel across with a constant velocity u_i . The equation of motion of the damped beam is given by the fifth order partial differential equation [16]

$$EI \frac{\partial^4 \psi(x,t)}{\partial x^4} + C_s I \frac{\partial^5 \psi(x,t)}{\partial x^4 \partial t} + \mu \frac{\partial^2 \psi(x,t)}{\partial t^2} - \mu R^0 \frac{\partial^4 \psi(x,t)}{\partial x^2 \partial t^2} + C \frac{\partial \psi(x,t)}{\partial t} - N \frac{\partial^2 \psi(x,t)}{\partial x^2} = P(x,t) \quad (1)$$

where E is the Young's Modulus, C_s is the damping due to strain resistance of the beam material, I is the moment of inertia, μ is the mass per unit length of the beam, C is the viscous damping coefficient, N is the axial force, $\psi(x, t)$ is the transverse displacement, x is the spatial coordinate and $P(x, t)$ is the transverse distributed load.

The boundary conditions of the structure under consideration is arbitrary and the initial conditions without any loss of generality is taken as

$$V(x,0) = 0 = \frac{\partial V(x,0)}{\partial t} \quad (2)$$

If the inertia effect of the moving load is considered, the load $P(x, t)$ takes the form [34]

$$P(x, t) = P_f(x, t) \left[1 - \frac{1}{g} \chi \psi(x, t) \right] \quad (3)$$

where $P_f(x, t)$ is the continuous moving force acting on the beam model given by

$$P_f(x, t) = \sum_{i=1}^N M_i g H[x - u_i t] \quad (4)$$

g is the acceleration due to gravity and χ is the convective acceleration operator defined as [16]

$$\chi = \frac{\partial^2}{\partial t^2} + 2u_i \frac{\partial^2}{\partial x \partial t} + u_i^2 \frac{\partial^2}{\partial x^2} \quad (5)$$

where the time t is assumed to be limited to that interval of time within the mass of the beam, that is

$$0 \leq u_i t \leq L \quad (6)$$

The geometry of moving load on a clamped-clamped beam is shown in Figure 1. Using Equations (3) to (5) in equation in Equation (1), one obtains

$$EI \frac{\partial^4 \psi(x, t)}{\partial x^4} + C_s I \frac{\partial^5 \psi(x, t)}{\partial x^4 \partial t} + \mu \frac{\partial^2 \psi(x, t)}{\partial t^2} - \mu R^0 \frac{\partial^4 \psi(x, t)}{\partial x^2 \partial t^2} + C \frac{\partial \psi(x, t)}{\partial t} - N \frac{\partial^2 \psi(x, t)}{\partial x^2} + \sum_{i=1}^N M_i H[x - u_i t] \left(\frac{\partial^2}{\partial t^2} + 2u_i \frac{\partial^2}{\partial x \partial t} + u_i^2 \frac{\partial^2}{\partial x^2} \right) \psi(x, t) = \sum_{i=1}^N M_i g H[x - u_i t] \quad (7)$$

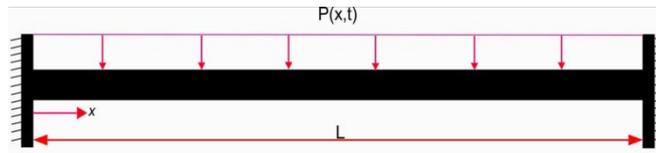


Figure 1. Geometry of moving load on a Clamped-Clamped beam

2.1 Solution Procedure

Equation (7) is a fifth order partial differential equation which has some coefficients which are not only variable but also singular. In this section, a general approach is developed in order to solve the initial value problem. The approach is employed to express the Heaviside function as a series form, remove the singularities in the governing equations and to reduce it to a sequence of second order ordinary differential equations with variable coefficients using the generalized finite integral transform. The resulting transformed differential equation is then simplified using modified Struble’s asymptotic technique. The generalized finite integral transform is defined by

$$\psi(m, t) = \int_0^L \psi(x, t) U_m(x) dx \tag{8}$$

with inverse

$$\psi(x, t) = \sum_{m=1}^{\infty} \frac{\mu}{\psi_m} \psi(m, t) U_m(x) \tag{9}$$

where

$$\psi_m = \int_0^L \mu U_m^2(x) dx \tag{10}$$

and $U_m(x)$ is any function chosen such that the pertinent boundary conditions are satisfied. An appropriate selection of functions for beam problems are beam mode shapes. Thus, for a uniform beam, the m -th normal mode of vibration

$$U_m(x) = \sin \frac{\lambda_m x}{L} + A_m \cos \frac{\lambda_m x}{L} + B_m \sinh \frac{\lambda_m x}{L} + C_m \cosh \frac{\lambda_m x}{L} \tag{11}$$

is chosen as a suitable kernel of the integral transform, where A_m, B_m, C_m are constants and the mode frequencies λ_m can be determined using appropriate classical boundary conditions.

2.2 Operational Simplification

Applying the generalized finite integral transform in Equation (8), Equation (7) becomes

$$\begin{aligned} &\psi_{tt}(m, t) + \left(\frac{C}{\mu} - \frac{\omega_m^2 C_s I}{EI} \right) \psi_t(m, t) + \omega_m^2 \psi(m, t) - \frac{R^0}{\mu L} \sum_{k=1}^{\infty} \frac{L}{\tau_k} \psi_{tt}(k, t) Q_B(k, m) - \frac{N}{\mu L} \sum_{k=1}^{\infty} \frac{1}{\tau_k} \psi(k, t) Q_B(k, m) \\ &+ \frac{M}{4} \sum_{k=1}^{\infty} \frac{1}{\tau_k} \psi_{tt}(k, t) Q_C(k, m) + \frac{M}{\pi} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\tau_k} \frac{\cos(2n+1)\pi u_i t}{2n+1} \psi_{tt}(k, t) Q_D(n, k, m) - \frac{M}{\pi} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\tau_k} \frac{\sin(2n+1)\pi u_i t}{2n+1} \\ &\psi_{tt}(k, t) Q_E(n, k, m) + \frac{Mu_i}{2\pi} \sum_{k=1}^{\infty} \frac{1}{\tau_k} \psi_t(k, t) Q_F(k, m) + \frac{2Mu_i}{\pi} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\tau_k} \frac{\cos(2n+1)\pi u_i t}{2n+1} \psi_t(k, t) Q_G(n, k, m) \\ &- \frac{2Mu_i}{\mu\pi} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\tau_k} \frac{\sin(2n+1)\pi u_i t}{2n+1} \psi_t(k, t) Q_H(n, k, m) - \frac{Mu_i^2}{4} \sum_{k=1}^{\infty} \frac{1}{\tau_k} \psi(k, t) Q_B(k, m) + \frac{Mu_i^2}{\pi} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\tau_k} \frac{\cos(2n+1)\pi u_i t}{2n+1} \\ &\psi(k, t) Q_I(n, k, m) - \frac{Mu_i^2}{\pi} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\sigma_k} \frac{\sin(2n+1)\pi u_i t}{2n+1} \psi(k, t) Q_J(n, k, m) = \frac{MgL}{\mu\lambda_m} \left[\cos \lambda_m + A_m \sin \lambda_m + \right. \\ &\left. B_m \cosh \lambda_m + C_m \sinh \lambda_m + \cos \frac{\lambda_m u_i t}{L} - A_m \sin \frac{\lambda_m u_i t}{L} - B_m \cosh \frac{\lambda_m u_i t}{L} - C_m \sinh \frac{\lambda_m u_i t}{L} \right] \end{aligned} \tag{12}$$

Rearranging Equation (12) gives

$$\begin{aligned} & \psi_{tt}(m, t) + \omega_m^2 \psi(m, t) - \frac{R^0}{L} \left[\sum_{k=1}^{\infty} \frac{L}{\tau_k} Q_B(k, m) \psi_{tt}(k, t) - \frac{L}{R^0} \left(\frac{C}{\mu} - \frac{\omega_m^2 C_s I}{EI} \right) \psi_t(m, t) - \frac{N}{\mu L R^0} \sum_{k=1}^{\infty} \frac{L}{\tau_k} Q_B(k, m) \psi(k, t) \right] \\ & + \varepsilon_0 \left\{ \sum_{k=1}^{\infty} \frac{1}{\tau_k} \left[\frac{1}{4} Q_C(k, m) \psi_{tt}(k, t) + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi u_i t}{2n+1} Q_D(n, k, m) \psi_{tt}(k, t) - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi u_i t}{2n+1} Q_E(n, k, m) \psi_{tt}(k, t) \right. \right. \\ & + \frac{u_i}{2\pi} Q_F(k, m) \psi_t(k, t) + \frac{2u_i}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi u_i t}{2n+1} Q_G(n, k, m) \psi_t(k, t) - \frac{2u_i}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi u_i t}{2n+1} Q_H(n, k, m) \psi_t(k, t) \\ & \left. + \frac{u_i^2}{4} Q_B(k, m) \psi(k, t) + \frac{u_i^2}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi u_i t}{2n+1} Q_I(n, k, m) \psi(k, t) - \frac{u_i^2}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi u_i t}{2n+1} Q_J(n, k, m) \psi(k, t) \right\} \\ & = \frac{P}{\mu \lambda_m} [\cos \lambda_m + A_m \sin \lambda_m + B_m \cosh \lambda_m + C_m \sinh \lambda_m + \cos \alpha u_i t - A_m \sin \alpha u_i t - B_m \cosh \alpha u_i t - C_m \sinh \alpha u_i t] \end{aligned} \tag{13}$$

where

$$\varepsilon_0 = \frac{M}{\mu L}, P = \frac{MgL}{\mu}, \alpha u_i = \frac{\lambda_m u_i}{L}, \omega_m^2 = \frac{EI \lambda_m^4}{\mu L^4} \tag{14}$$

$$\begin{aligned} Q_A(k, m) &= \int_0^L U_k^{iv}(x) U_m(x) dx; Q_B(k, m) = \int_0^L U_k''(x) U_m(x) dx; Q_C(k, m) = \int_0^L U_k(x) U_m(x) dx \\ Q_D(n, k, m) &= \int_0^L \sin(2n+1)\pi x U_k(x) U_m(x) dx; Q_E(n, k, m) = \int_0^L \cos(2n+1)\pi x U_k(x) U_m(x) dx \\ Q_F(k, m) &= \int_0^L U_k'(x) U_m(x) dx; Q_G(n, k, m) = \int_0^L \sin(2n+1)\pi x U_k'(x) U_m(x) dx \\ Q_H(n, k, m) &= \int_0^L \cos(2n+1)\pi x U_k'(x) U_m(x) dx; Q_I(n, k, m) = \int_0^L \sin(2n+1)\pi x U_k''(x) U_m(x) dx \\ Q_J(n, k, m) &= \int_0^L \cos(2n+1)\pi x U_k''(x) U_m(x) dx; \tau_k(x) = \int_0^L U_k^2(x) dx \end{aligned} \tag{15}$$

Equation (13) is the transformed equation governing the problem of a Rayleigh beam on a constant elastic foundation. This is a coupled non-homogeneous second order ordinary differential equation holds for all variants of the classical boundary conditions. In what follows, two cases of Equation (13) are considered.

3. SOLUTION OF THE TRANSFORMED GOVERNING EQUATION

3.1 Case I: Rayleigh Beam Traversed by a Moving Force

The differential equation describing the behaviour of a uniform Rayleigh beam on an elastic foundation to a moving force moving at constant velocity may be obtained from Equation (13) by setting $\varepsilon_0 = 0$. In this case, one obtains

$$\begin{aligned} & \psi_{tt}(m, t) + \omega_m^2 \psi(m, t) - \frac{R^0}{L} \left[\sum_{k=1}^{\infty} \frac{L}{\tau_k} Q_B(k, m) \psi_{tt}(k, t) - \frac{L}{R^0} \left(\frac{C}{\mu} - \frac{\omega_m^2 C_s I}{EI} \right) \psi_t(m, t) - \frac{N}{\mu L R^0} \sum_{k=1}^{\infty} \frac{L}{\tau_k} Q_B(k, m) \psi(k, t) \right] \\ & = \frac{P}{\mu_0 \lambda_m} [\cos \lambda_m + A_m \sin \lambda_m + B_m \cosh \lambda_m + C_m \sinh \lambda_m + \cos \alpha u_i t - A_m \sin \alpha u_i t - B_m \cosh \alpha u_i t - C_m \sinh \alpha u_i t] \end{aligned} \tag{16}$$

The analytical solution to Equation (16) is not possible. Though the equation yields readily to numerical technique, an analytical approximate method is desirable as the solution obtained often shed light on vital information about the vibrating system. To this end, we are going to use a modification of the asymptotic method due to Struble's. For this purpose, Equation (16) is rearranged to take the form

$$\begin{aligned} & \psi_{tt}(m, t) + \omega_{mf}^2 \psi(m, t) + \frac{\varepsilon_1^* F_1}{1 - \varepsilon_1^* L Q_B(m, m)} \psi_t(m, t) - \frac{\varepsilon_1^*}{1 - \varepsilon_1^* L Q_B(m, m)} \left[\sum_{\substack{k=1 \\ k \neq m}}^{\infty} L Q_B(k, m) \psi_{tt}(k, t) \right. \\ & \left. - F_2 \sum_{\substack{k=1 \\ k \neq m}}^{\infty} L Q_B(k, m) \psi_{tt}(k, t) \right] = \frac{P}{\lambda_m \mu [1 - \varepsilon_1^* L Q_B(m, m)]} [\phi_m + \cos \alpha u_i t - A_m \sin \alpha u_i t - B_m \cosh \alpha u_i t - C_m \sinh \alpha u_i t] \end{aligned} \tag{17}$$

where

$$\omega_{mf}^2 = \frac{\varepsilon_1^* L Q_B(m, m) + \omega_m^2}{1 - \varepsilon_1^* L Q_B(m, m)}, \varepsilon_1^* = \frac{R^0}{L}, F_1 = \frac{L}{R^0} \left(\frac{C}{\mu} - \frac{\omega_m^2 C_s I}{EI} \right), F_2 = \frac{N}{\mu L R^0} \tag{18}$$

$$Q_A(m, m) = Q_A(k, m)|_{k=m}, Q_B(m, m) = Q_B(k, m)|_{k=m} \tag{19}$$

$$\phi_m = \cos \lambda_m + A_m \sin \lambda_m + B_m \cosh \lambda_m + C_m \sinh \lambda_m \tag{20}$$

By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of axial force N . An equivalent free system operator defined by the modified frequency then replaces Equation (17). Thus, we set the right hand side of Equation (17) to zero and considered a parameter $\eta \ll 1$ for any arbitrary ratio ε^* , defined as

$$\eta = \frac{\varepsilon_1^*}{1 + \varepsilon_1^*} \tag{21}$$

so that

$$\varepsilon_1^* = \eta + O(\eta^2) \tag{22}$$

Substituting Equation (22) in the homogeneous part of Equation (17), one obtains

$$\begin{aligned} &\psi_{tt}(m, t) + \eta F_1 [1 + \eta L Q_B(m, m)] \psi_t(m, t) + \omega_{mf}^2 [1 + \eta L Q_B(m, m)] \psi(m, t) - \\ &\eta [1 + \eta L Q_B(m, m)] \left[\sum_{\substack{k=1 \\ k \neq m}}^{\infty} L Q_B(k, m) \psi_{tt}(k, t) - F_2 \sum_{\substack{k=1 \\ k \neq m}}^{\infty} Q_A(k, t) \psi(k, t) \right] = 0 \end{aligned} \tag{23}$$

When η is set to zero in Equation (23) a situation corresponding to the case in which the axial force effect is regarded as negligible is obtained, then the solution of Equation (23) becomes

$$\psi(m, t) = \beta(m, t) \cos[\omega_{mf} t - \alpha(m, t)] \tag{24}$$

where $\beta(m, t)$, ω_{mf} and $\alpha(m, t)$ are constants. Furthermore as $\eta \ll 1$, Struble's technique requires that the asymptotic solutions of the homogeneous part of Equation (23) be of the form

$$\psi(m, t) = \beta(m, t) \cos[\omega_{mf} t - \alpha(m, t)] + \eta \psi_1 + O(\eta^2) \tag{25}$$

where $\beta(m, t)$ and $\alpha(m, t)$ are slowly varying functions of time. To obtain the modified frequency, Equations (25) and its derivatives are substituted into Equation (23) and neglecting terms which do not contribute to the variational equations, one obtains

$$\begin{aligned} &- 2\dot{\beta}(m, t) \omega_{mf} \sin[\omega_{mf} t - \alpha(m, t)] + 2\beta(m, t) \omega_{mf} \dot{\alpha}(m, t) \cos[\omega_{mf} t - \alpha(m, t)] \\ &- \beta(m, t) \omega_{mf}^2 \cos[\omega_{mf} t - \alpha(m, t)] - \eta F_1 \beta(m, t) \omega_{mf} \sin[\omega_{mf} t - \alpha(m, t)] \\ &+ \beta(m, t) \omega_{mf}^2 \cos[\omega_{mf} t - \alpha(m, t)] + \eta L Q_B(m, m) \beta(m, t) \omega_{mf}^2 \cos[\omega_{mf} t - \alpha(m, t)] = 0 \end{aligned} \tag{26}$$

retaining terms to $O(\eta)$ only.

The variational equations are obtained by equating the coefficients of $\sin(\omega_{mf} - \alpha(m, t))$ and $\cos(\omega_{mf} - \alpha(m, t))$ on both sides of Equation (26), we obtain

$$- 2\dot{\beta}(m, t) \omega_{mf} = \eta F_1 \beta(m, t) \omega_{mf} \tag{27}$$

and

$$2\beta(m, t) \omega_{mf} \dot{\alpha}(m, t) + \eta L Q_B(m, m) \beta(m, t) \omega_{mf}^2 = 0 \tag{28}$$

Solving Equations (27) and (28) simultaneously gives

$$\beta(m, t) = A_x e^{-\theta} \tag{29}$$

and

$$\alpha(m, t) = \frac{1}{2} \eta L Q_B(m, m) \omega_{mf} t + \xi_m \tag{30}$$

where

$$\theta = \eta F_1 \omega_{mf} \tag{31}$$

Therefore, when the effect of the axial force is considered, the first approximation to the homogeneous system is

$$\tilde{\psi}(m, t) = A_x e^{-\theta t} \cos[\gamma_{aj} t - \xi_m] \tag{32}$$

where

$$\gamma_{aj} = \omega_{mf} \left[1 - \frac{\eta}{2} (LQ_B(m, m)) \right] \tag{33}$$

represents the modified natural frequency due to the effect of axial force. It is observed that when $\eta = 0$, we recover the frequency of the moving force problem when the axial force effect of the beam is considered negligible. Thus, to solve the non-homogeneous Equation (17), the differential operator which act on $\tilde{\psi}(m, t)$ and $\tilde{\psi}(k, t)$ are replaced by the equivalent free system operator defined by the modified frequency, γ_{aj} , thus, using Equation (33), the homogeneous part of Equation (17) can be written as

$$\tilde{\psi}_{tt}(m, t) + \gamma_{aj}^2 \tilde{\psi}(m, t) = 0 \tag{34}$$

Hence, the entire equation of Equation (17) takes the form of

$$\tilde{\psi}_{tt}(m, t) + \gamma_{aj}^2 \tilde{\psi}(m, t) = \frac{P}{\lambda_m} [\phi_m + \cos \alpha u_i t - A_m \sin \alpha u_i t - B_m \cosh \alpha u_i t - C_m \sinh \alpha u_i t] \tag{35}$$

Thus, taking the Laplace transform of (35), one obtains the algebraic equation

$$(s^2 + \gamma_{aj}^2) \tilde{\psi}_{tt}(m, t) = \frac{P}{\lambda_m} \left[\frac{\phi_m}{s} + \frac{s}{s^2 + (\alpha u_i)^2} - \frac{A_m}{s^2 + (\alpha u_i)^2} - \frac{B_m}{s^2 - (\alpha u_i)^2} - \frac{C_m}{s^2 - (\alpha u_i)^2} \right] \tag{36}$$

Equation (35) can further be simplified as

$$\tilde{\psi}(m, s) = \frac{P}{\lambda_m} [\phi_m F_1 + F_2 - A_m F_3 - B_m F_4 - C_m F_5] \tag{37}$$

where

$$F_1 = \frac{Q(m, c)}{s(s^2 + \gamma_{aj}^2)} \tag{38}$$

$$F_2 = \frac{s}{(s^2 + (\alpha u_i)^2)(s^2 + \gamma_{aj}^2)} \tag{39}$$

$$F_3 = \frac{A_m \alpha u_i}{(s^2 + (\alpha u_i)^2)(s^2 + \gamma_{aj}^2)} \tag{40}$$

$$F_4 = \frac{B_m \alpha u_i}{(s^2 - (\alpha u_i)^2)(s^2 + \gamma_{aj}^2)} \tag{41}$$

$$F_5 = \frac{C_m \alpha u_i}{(s^2 - (\alpha u_i)^2)(s^2 + \gamma_{aj}^2)} \tag{42}$$

Taking the Laplace inversion of Equation (37), one obtains

$$\begin{aligned} \tilde{\psi}(m, t) = \frac{P}{\lambda_m} & \left\{ \phi_m \frac{(1 - \cos \gamma_{aj} t)}{\gamma_{aj}} + \frac{\cos \alpha u_i t - \cos \gamma_{aj} t}{\gamma_{aj}^2 - \alpha u_i^2} - \frac{A_m}{\gamma_{aj}(\gamma_{aj}^2 - \alpha u_i^2)} [\gamma_{aj} \sin \alpha u_i t - \alpha u_i \sin \gamma_{aj} t] \right. \\ & + \frac{B_m}{\alpha u_i^4 - \gamma_{aj}^4} \left[\gamma_{aj} \alpha u_i \sin 2\gamma_{aj} t \sinh \alpha u_i t + \gamma_{aj}^2 \cos 2\gamma_{aj} t \cosh \alpha u_i t - \alpha u_i^2 \cosh \alpha u_i t + \frac{(\alpha u_i^2 - \gamma_{aj}^2) \cos \gamma_{aj} t}{\gamma_{aj}} \right] \\ & \left. - \frac{C_m}{\alpha u_i^4 - \gamma_{aj}^4} \left[\gamma_{aj} \alpha u_i \sin 2\gamma_{aj} t \sinh \alpha u_i t + \cosh \alpha u_i t + \alpha u_i^2 \sinh \alpha u_i t + \cos 2\gamma_{aj} t - \frac{(\alpha u_i^2 - \gamma_{aj}^2) \sin \gamma_{aj} t}{\gamma_{aj}} \right] \right\} \tag{43} \end{aligned}$$

Thus in view of Equation (9), Equation (43) becomes

$$\begin{aligned} \tilde{\psi}(x, t) = & \frac{1}{\tau_m} \sum_{m=1}^{\infty} \frac{\mu P}{\psi_m \lambda_m} \left\{ \phi_m \frac{(1 - \cos \gamma_{aj} t)}{\gamma_{aj}} + \frac{\cos \alpha u_i t - \cos \gamma_{aj} t}{\gamma_{aj}^2 - \alpha u_i^2} - \frac{A_m}{\gamma_{aj} (\gamma_{aj}^2 - \alpha u_i^2)} \left[\gamma_{aj} \sin \alpha u_i t - \alpha u_i \sin \gamma_{aj} t \right] \right. \\ & + \frac{B_m}{\alpha u_i^4 - \gamma_{aj}^4} \left[\gamma_{aj}^2 \cos 2\gamma_{aj} t \cosh \alpha u_i t - \alpha u_i^2 \cosh \alpha u_i t + \gamma_{aj} \alpha u_i \sin 2\gamma_{aj} t \sinh \alpha u_i t + \frac{(\alpha u_i^2 - \gamma_{aj}^2) \cos \gamma_{aj} t}{\gamma_{aj}} \right] \\ & \left. - \frac{C_m}{\alpha u_i^4 - \gamma_{aj}^4} \left[\cosh \alpha u_i t - \cos 2\gamma_{aj} t + \alpha u_i^2 \sinh \alpha u_i t + \gamma_{aj} \alpha u_i \sin 2\gamma_{aj} t \sinh \alpha u_i t - \frac{\alpha u_i (\alpha u_i^2 - \gamma_{aj}^2) \sin \gamma_{aj} t}{\gamma_{aj}} \right] \right\} \cdot (44) \\ & \left[\sin \frac{\lambda_m x}{L} + A_m \cos \frac{\lambda_m x}{L} + B_m \sinh \frac{\lambda_m x}{L} + C_m \cosh \frac{\lambda_m x}{L} \right] \end{aligned}$$

Equation (44) represents the transverse displacement response to a moving force of a prestressed uniform Bernoulli-Euler beam resting on constant Winkler elastic foundation and having arbitrary edge supports.

3.2 Case II: Rayleigh Beam Traversed by a Moving Mass

If the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving load is not negligible. Thus $\epsilon_0 \neq 0$ and one is required to solve the entire Equation (13) when no term of the coupled differential equation is neglected. This is termed moving mass problem.

Evidently, an exact analytical solution to Equation (13) is not possible. Thus a modification of Struble’s technique discussed in Oni [34] is employed. Consequently, Equation (13) is rearranged to take the form

$$\begin{aligned} \tilde{\psi}_{tt}(m, t) + \frac{\epsilon_0 P_2(m, n, t)}{1 + \epsilon_0 P_1(m, n, t)} \tilde{\psi}_t(m, t) + \frac{\gamma_{aj}^2 + \epsilon_0 P_3(m, n, t)}{1 + \epsilon_0 P_1(m, n, t)} \tilde{\psi}(m, t) \\ = \frac{P}{\lambda_m (1 + \epsilon_0 P_1(m, n, t))} \left[\phi_m + \cos \alpha u_i t - A_m \sin \alpha u_i t - B_m \cosh \alpha u_i t - C_m \sinh \alpha u_i t \right] \end{aligned} \quad (45)$$

where

$$\begin{aligned} P_1(m, n, t) &= \frac{1}{4} Q_C(m, m) + \frac{1}{\pi} \sum_{n=0}^{\infty} \Delta_{11}(t) Q_D(n, m, m) - \frac{1}{\pi} \sum_{n=0}^{\infty} \Delta_{22}(t) Q_E(n, m, m) \\ P_2(m, n, t) &= \frac{u_i}{2} Q_F(m, m) + \frac{2u_i}{\pi} \sum_{n=0}^{\infty} \Delta_{11}(t) Q_G(n, m, m) - \frac{2u_i}{\pi} \sum_{n=0}^{\infty} \Delta_{22}(t) Q_H(n, m, m) \\ P_3(m, n, t) &= \frac{u_i}{2} Q_B(m, m) + \frac{2u_i^2}{\pi} \sum_{n=0}^{\infty} \Delta_{11}(t) Q_I(n, m, m) - \frac{2u_i^2}{\pi} \sum_{n=0}^{\infty} \Delta_{22}(t) Q_J(n, m, m) \\ \epsilon_0 &= \frac{M}{\mu L}, \quad \Delta_{11}(t) = \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi u_i t}{2n+1}, \quad \Delta_{22}(t) = \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi u_i t}{2n+1} \end{aligned} \quad (46)$$

As in the previous case, the homogeneous part is first considered and a modified frequency corresponding to the frequency of the free system due to the presence of moving mass is sought. An equivalent free system operator defined by the modified frequency then replaces Equation (46). Thus, the right-hand side of Equation (46) is said to zero and parameter

$$\lambda = \frac{\epsilon_0}{1 + \epsilon_0} \quad (47)$$

Thus

$$\epsilon_0 = \lambda + O(\lambda^2) \quad (48)$$

which implies

$$\frac{1}{1 + \epsilon_0 P_1(m, n, t)} = 1 - \epsilon_0 P_1(m, n, t) + O(\lambda^2) \quad (49)$$

where

$$|\epsilon_0 P_1(m, n, t)| < 1 \quad (50)$$

which implies that all the coefficients of $\tilde{\psi}(m, t)$ and its derivatives in Equation (45) can be written in terms of the parameter λ . When λ is set to zero in Equation (45), a situation corresponding to the case in which the axial force effect of the mass of the system is regarded as negligible is obtained. In such a case, the solution is of the form

$$\tilde{\psi}(m, t) = C(m, t) \cos[\gamma_{aj}t - \xi_m] + \lambda \psi_1 + O(\lambda^2) \tag{51}$$

where $C(m, t)$ and ξ_m are slowly vary functions of time.

Following the same arguments with those in the previous section using the Struble’s technique, we obtain

$$\begin{aligned} & -2\dot{C}(m, t)\gamma_{aj} \sin[\gamma_{aj}t - \phi(m, t)] + 2C(m, t)\gamma_{aj}\dot{\phi}(m, t) \cos[\gamma_{aj}t - \phi(m, t)] - \\ & C(m, t)\gamma_{aj}\lambda P_2(m, n, t) \sin[\gamma_{aj}t - \phi(m, t)] - \lambda P_1(m, n, t)C(m, t)\gamma_{aj}^2 \cos[\gamma_{aj}t - \phi(m, t)] + \\ & \lambda P_3(m, n, t)C(m, t) \cos[\gamma_{aj}t - \phi(m, t)] = 0 \end{aligned} \tag{52}$$

retaining terms to $O(\lambda)$ only. The variational equations are obtained from Equation (52) as

$$-2\dot{C}(m, t)\gamma_{aj} - C(m, t)\gamma_{aj}\lambda P_2(m, n, t) = 0 \tag{53}$$

and

$$2C(m, t)\gamma_{aj}\dot{\phi}(m, t) + \lambda P_3(m, n, t)C(m, t) = \lambda P_1(m, n, t)C(m, t)\gamma_{aj}^2 \tag{54}$$

Solving Equations (53) and (54) simultaneously, one obtains

$$C(m, t) = A_0 e^{-\frac{1}{2}P_1(m, n, t)t} \tag{55}$$

and

$$\phi(m, t) = \frac{\lambda}{2\gamma_{aj}} \left[\gamma_{aj}^2 P_1(m, n, t) + P_3(m, n, t) \right] t + \beta_m \tag{56}$$

where β_m is a constant. The first approximation to the homogeneous system is given as

$$\tilde{\psi}(m, t) = A_0 e^{-\frac{1}{2}P_1(m, n, t)t} \cos[\gamma_{bj}t - \beta_m] \tag{57}$$

where

$$\gamma_{bj} = \gamma_{aj} \left\{ 1 - \frac{\lambda}{2} \left[P_1(m, n, t) + \frac{P_3(m, n, t)}{\gamma_{aj}^2} \right] \right\} \tag{58}$$

Thus, to solve the nonhomogeneous Equation (45), the differential equation which acts on $\psi(m, t)$ and $\psi(k, t)$ is replaced by the equivalent free system operator defined by the modified frequency γ_{bj} . We then have

$$\begin{aligned} \tilde{\psi}_{tt}(m, t) + \gamma_{bj}^2 \tilde{\psi}(m, t) = P_g \left[-\cos \lambda_m + A_m \sin \lambda_m + B_m \cosh \lambda_m + C_m \sinh \lambda_m + \right. \\ \left. \cos \alpha u_i t - A_m \sin \alpha u_i t - B_m \cosh \alpha u_i t - C_m \sinh \alpha u_i t \right] \end{aligned} \tag{59}$$

where

$$P_g = \frac{MgL}{\lambda_m} \tag{60}$$

It is noticed that Equation (59) is analogous to Equation (35). Therefore, when Equation (59) is solved in conjunction with the initial conditions, one obtains the expression for $\psi(m, t)$ and in view of Equation (9), one obtains

$$\begin{aligned}
 \tilde{\psi}(x, t) = \sum_{m=1}^{\infty} \frac{\mu P g}{\psi_m} & \left\{ \phi_m \frac{(1 - \cos \gamma_{bj} t)}{\gamma_{bj}} + \frac{\cos \alpha u_i t - \cos \gamma_{bj} t}{\gamma_{bj}^2 - \alpha u_i^2} - A_m \left[\frac{\gamma_{bj} \sin \alpha u_i t - \alpha u_i \sin \gamma_{bj} t}{\gamma_{bj} (\gamma_{bj}^2 - \alpha u_i^2)} \right] + \right. \\
 & \frac{B_m}{\alpha u_i^4 - \gamma_{bj}^4} \left[\gamma_{bj}^2 \cos 2\gamma_{bj} t \cosh \alpha u_i - \alpha u_i^2 \cosh \alpha u_i t + \gamma_{bj} \alpha u_i \sin 2\gamma_{bj} t \sinh \alpha u_i t + \right. \\
 & \left. \left. \frac{(\alpha u_i^2 - \gamma_{bj}^2) \cos \gamma_{bj} t}{\gamma_{bj}} \right] - \frac{C_m}{\alpha u_i^4 - \gamma_{bj}^4} \left[\cosh \alpha u_i t - \cos 2\gamma_{bj} t + \alpha u_i^2 \sinh \alpha u_i t + \right. \right. \\
 & \left. \left. \gamma_{bj} \alpha u_i \sin 2\gamma_{bj} t - \frac{(\alpha u_i^2 - \gamma_{bj}^2) \sin \gamma_{bj} t}{\gamma_{bj}} \right] \right\} \left[\sin \frac{\lambda_m x}{L} + A_m \cos \frac{\lambda_m x}{L} + B_m \sinh \frac{\lambda_m x}{L} + \right. \\
 & \left. C_m \cosh \frac{\lambda_m x}{L} \right]
 \end{aligned} \tag{61}$$

Equation (61) represents the transverse displacement response to a moving mass of a Rayleigh beam resting on constant Winkler elastic foundation and having arbitrary edge supports.

3.3 Analysis of the Solution

Next, the phenomenon of resonance is examined. Equation (44) clearly shows that the beam on an elastic foundation and traversed by a moving force reaches a state of resonance whenever

$$\gamma_{aj} = \alpha u_i \tag{62}$$

while Equation (61) shows that the same beam under the action of moving mass experiences resonance effect whenever

$$\gamma_{bj} = \alpha u_i \tag{63}$$

where

$$\gamma_{bj} = \gamma_{aj} \left\{ 1 - \frac{\lambda}{2} \left[P_1(m, n, t) + \frac{P_3(m, n, t)}{\gamma_{aj}^2} \right] \right\} \tag{64}$$

Equations (62) and (63) imply that

$$\gamma_{bj} = \gamma_{aj} \left\{ 1 - \frac{\lambda}{2} \left[P_1(m, n, t) + \frac{P_3(m, n, t)}{\gamma_{aj}^2} \right] \right\} = \alpha u_i \tag{65}$$

Consequently from Equations (62) and (63) show that for the same natural frequency, the critical speed and the natural frequency for the same system of a uniform Rayleigh beam traversed by a moving force is greater than that of the same system traversed by a moving mass, for all variants of classical boundary conditions. Thus, for some natural frequency of the beam, resonance is reached earlier when consider the moving mass system than when we consider the moving force system.

4. ILLUSTRATIVE EXAMPLE

In this section, we shall illustrate the foregoing analysis by one practical example. Particularly we shall consider classical boundary conditions such as clamped-clamped end conditions.

4.1 Clamped-Clamped End Conditions

In this case, both deflection and slope vanish. Thus

$$\psi(0, t) = 0 = \psi(L, t), \quad \frac{\partial \psi(0, t)}{\partial x} = 0 = \frac{\partial \psi(L, t)}{\partial x} \tag{66}$$

and for normal modes

$$U_m(0) = 0 = U_m(L), \quad \frac{\partial U_m(0)}{\partial x} = 0 = \frac{\partial U_m(L)}{\partial x} \tag{67}$$

which implies that

$$U_k(0) = 0 = U_k(L), \quad \frac{\partial U_k(0)}{\partial x} = 0 = \frac{\partial U_k(L)}{\partial x} \tag{68}$$

Applying Equation (67) to Equation (11) yields

$$A_m = \frac{\sinh \lambda_m - \sin \lambda_m}{\cos \lambda_m - \cosh \lambda_m} = \frac{\cos \lambda_m - \cosh \lambda_m}{\sin \lambda_m + Ssnh \lambda_m} = -C_m \quad \text{and} \quad B_m = -1 \tag{69}$$

The frequency equation becomes

$$\cos \lambda_m \cosh \lambda_m = 1 \tag{70}$$

which is termed the frequency equation of the dynamical problem, such that [2]

$$\lambda_1 = 4.73004, \quad \lambda_2 = 7.85320, \quad \lambda_3 = 10.99561 \tag{71}$$

Using Equations (69) and (71) in Equations (44) and (61), one obtains the displacement response respectively to a moving force and a moving mass of a clamped-clamped Rayleigh beam resting on a Winkler elastic foundation.

5. NUMERICAL RESULTS AND DISCUSSION

In this section, calculations of practical interests in dynamics of structures and engineering design are presented for the illustrative example considered. An elastic clamped-clamped Rayleigh beam of length 12.192 m has been considered. Furthermore, it is assumed that the moving load travels at the constant velocity 8.123 m/s. The values of E, I and μ are chosen as 3.1×10^{10} N/m², 2.87698×10^{-3} m⁴ and 2758.291 kg/m respectively. Figure 2 displays the displacement response of an elastic beam under a moving distributed force for various values of axial force N . Analyses were carried out for fixed values of damping due to strain resistance C_s , rotatory inertia R^0 and the viscous damping coefficient C . It is observed that as N increases, the beam transverse displacement reduces. Similar results are obtained when the beam is subjected to moving distributed masses as shown in Figure 3. In Figure 4, for various time t , the displacement of the beam under a moving distributed force for various values of damping due to strain resistance C_s and fixed values of N, R^0 and C are plotted. Results show that higher values of C_s reduce the response amplitudes of the elastic beam. Similar results are obtained when the beam is subjected to moving distributed masses as in Figure 5. Also, Figure 6 shows that for various values of rotatory inertia R^0 and fixed values of axial force $N = 200000$, damping due to strain resistance $C_s = 15$ and the damping coefficient $C = 0.5$, higher values of rotatory inertia reduce the deflection profile of the vibrating structure. The same result is obtained when the clamped-clamped beam is traversed by a distributed mass as indicated in Figure 7. It is observed in Figures 8 and 9 that as the value of the viscous damping coefficient C increase, the deflection amplitude of the clamped-clamped beam decrease for both cases of moving distributed force and moving distributed mass respectively.

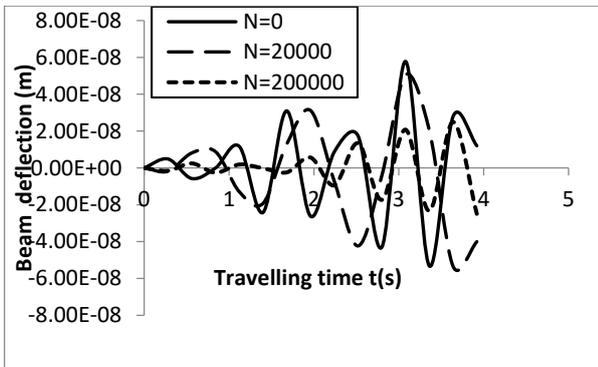


Figure 2. Displacement response of moving force for C-C beam for various $N, C_s = 7.5, R^0 = 150, C = 0.5$

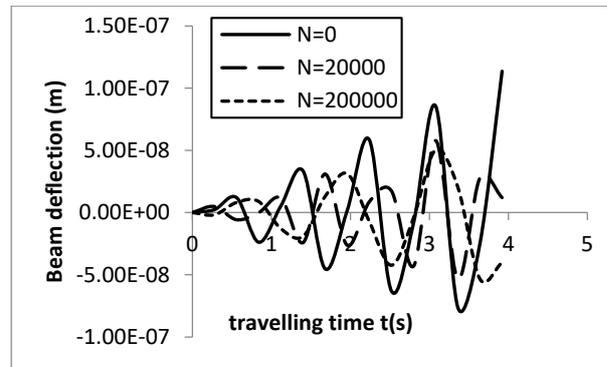


Figure 3. Displacement response of moving mass for C-C beam for various $N, C_s = 7.5, R^0 = 150, C = 0.5$

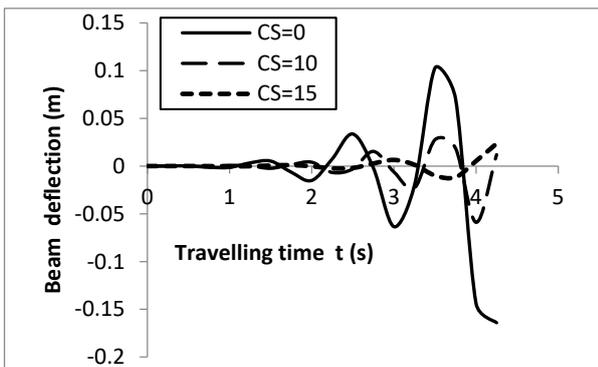


Figure 4. The deflection profile of moving force for C-C beam for various $C_s, N = 20000, R^0 = 150, C = 0.5$

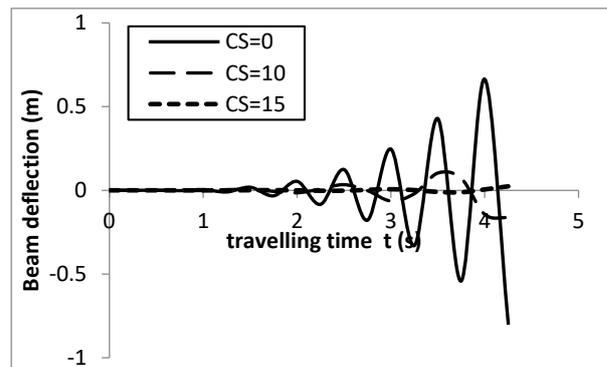


Figure 5. The deflection profile of moving mass for C-C beam for various $C_s, N = 20000, R^0 = 150, C = 0.5$

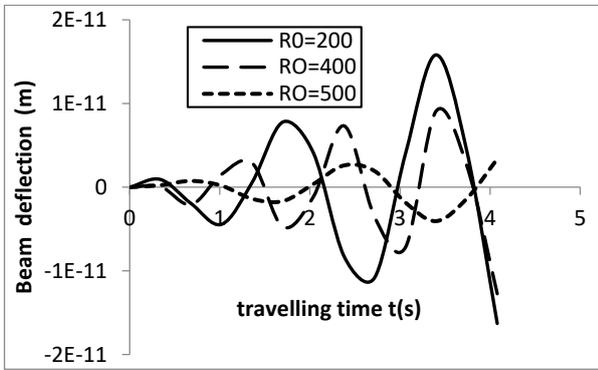


Figure 6. Displacement response of moving force for C-C beam for various R^0 , $N = 20000$, $C_s = 7.5$, $C = 0.5$

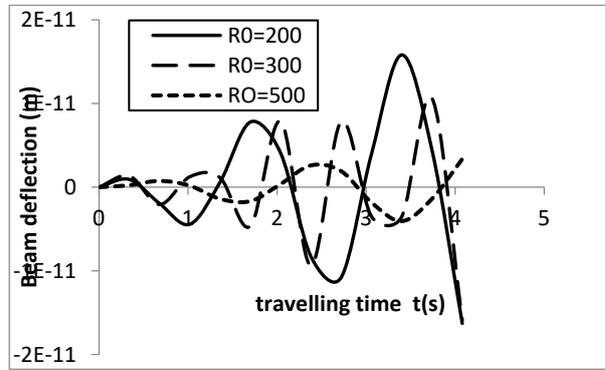


Figure 7. Displacement response of moving mass for C-C beam for various R^0 , $N = 20000$, $C_s = 7.5$, $C = 0.5$

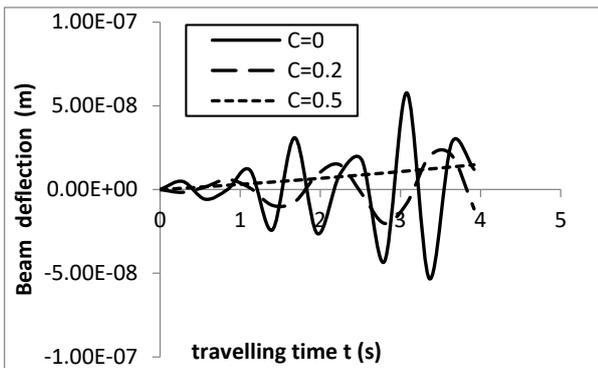


Figure 8. The deflection profile of moving force for C-C beam for various C , $N = 20000$, $R^0 = 150$, $C_s = 7.5$

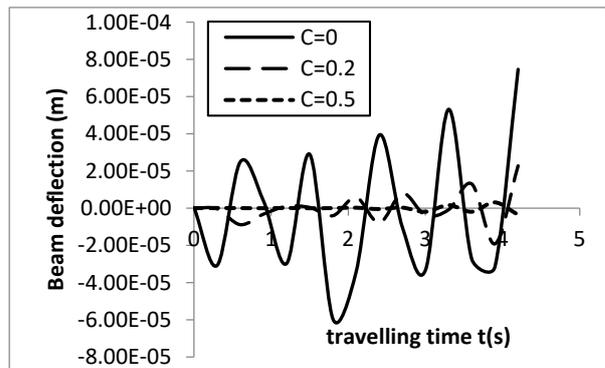


Figure 9. The deflection profile of moving mass for C-C beam for various C , $N = 20000$, $R^0 = 150$, $C_s = 7.5$

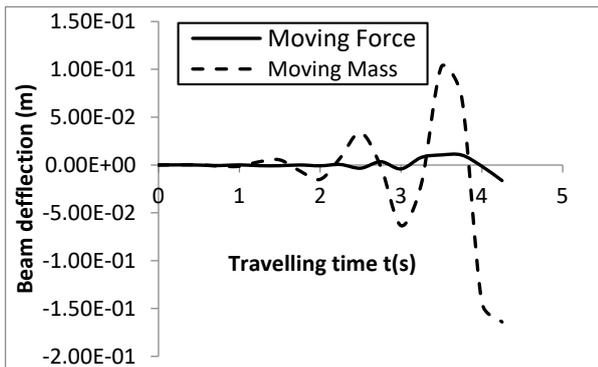


Figure 10. Comparison of moving force and moving mass cases of a clamped-clamped beam

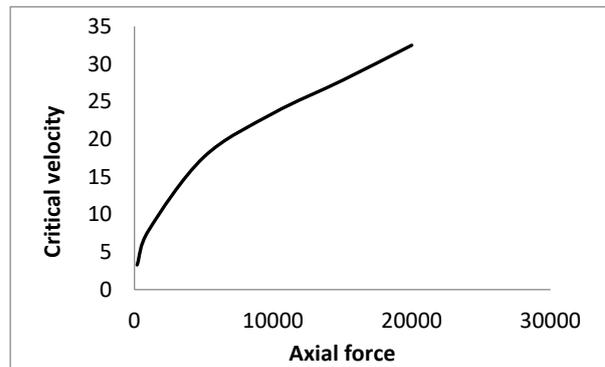


Figure 11. Variation of the critical velocity against the axial force

Figure 10 depicts the comparison of the displacement response of the moving distributed force and moving distributed mass cases of the clamped-clamped beam for fixed values of $N = 200000$, $C_s = 15$, $R^0 = 200$ and $C = 0.5$. Clearly, the response amplitude of the moving distributed mass system is greater than that of the moving distributed force system. In Figure 11, the relationship between the critical velocity and axial force is displayed. It is shown that as the axial force increases the critical velocity of the system also increases. Similar result is obtained in Figure 12 which depicts that as the viscous damping coefficient increases the critical velocity of the system also increases. For fixed values of N and C , Figure 13 clearly shows that as the values of rotatory inertia increases, the critical velocity of the system decreases. In Figure 14, it is clearly shown that increase in the mass ratio increases the critical velocity of the dynamical system. Figure 15 depicts that as the values of modified natural frequency of the system increases and for fixed values of other parameters, the viscous damping coefficient of the beam also increases. Similar result is obtained in Figure 16 which show that as the values of the mass ratio increases the modified natural frequency of the system increases. In Figure 17, it is shown that as the velocity of the traversing load increases the modified natural frequency of the system also increases.

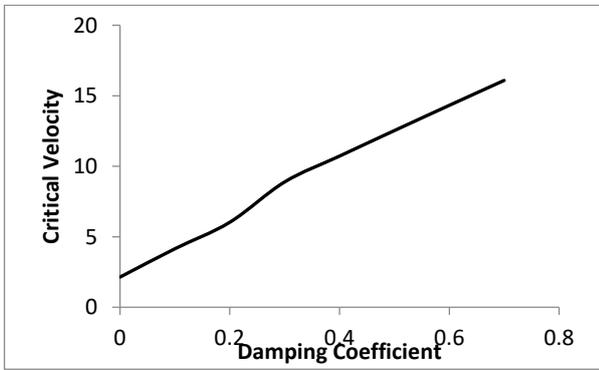


Figure 12. Variation of critical velocity versus damping coefficient

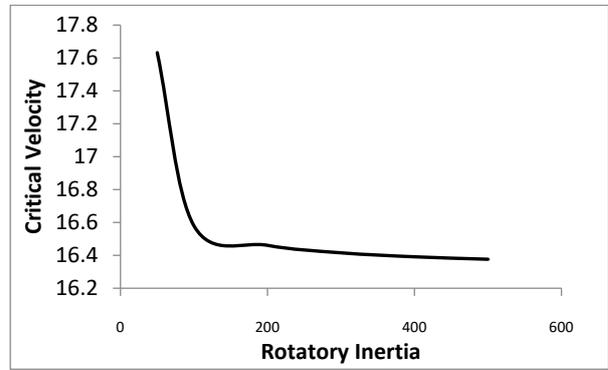


Figure 13. Variation of critical velocity versus rotatory inertia

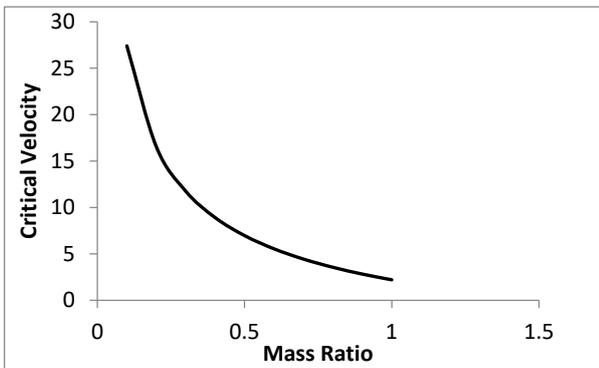


Figure 14. Variation of the critical velocity against the mass ratio

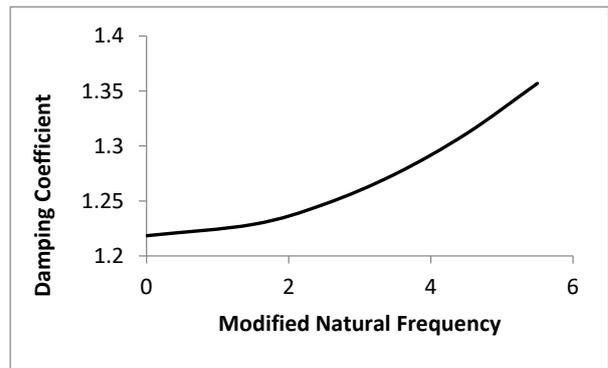


Figure 15. Variation of the critical velocity against modified natural frequency

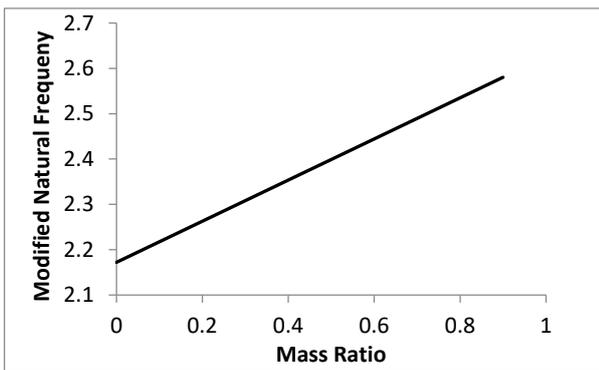


Figure 16. Variation of the modified natural frequency versus the mass ratio

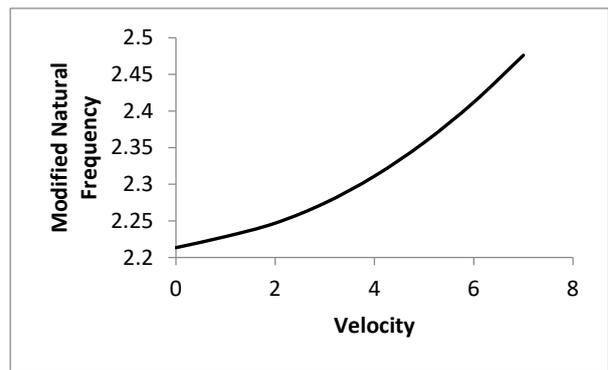


Figure 17. Variation of the modified natural frequency versus velocity of the moving load

6. CONCLUSION

This paper presents an analytical solution of the transverse displacement of a clamped-clamped Rayleigh beam under moving distributed loads. An approximate method of solution has been employed to treat the fifth order partial differential equations of motion describing the dynamic interactions of the continuous system and the moving sub-system. The modified Struble's technique and the method of generalized finite integral transform are employed to obtain the closed form solution of the transformed equation for both cases of moving distributed force and moving distributed mass problems. The analyses show that the moving force solution is not always an upper bound for the accurate solution of the moving mass problem and as the axial force increases, the response amplitudes of the beam decrease for both moving distributed force and moving distributed mass problem. The displacements of the Rayleigh beam decrease with increase in the various values of viscous damping coefficient and rotatory inertia. It is also found that as the values of axial force and viscous damping coefficient increase, the critical velocities increase indicating a safer dynamical system. Furthermore, for the natural frequency, the critical velocity for the system consisting of a clamped-clamped Rayleigh beam and traversed by a distributed force moving with a uniform velocity is greater than that of the distributed moving mass. Thus, resonance is reached earlier in distributed moving mass

system than in the distributed moving force. Finally, the results in this work are verified and found to be in agreement with what obtained in literature [35, 36]. Hence the method employed in this work is accurate and the solutions are convergent.

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