APPLICATIONS OF
MODELLING AND SIMULATION
http://arqiipubl.com/ams eISSN 2600-8084 VOL 5, 2021, 44-52

Numerical Simulation of a Thermo-piezoelectric Contact Issue with Tresca's Friction Law

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Submitted 18 September 2020; Revised 23 October 2020; Accepted 28 October 2020.

Abstract: A frictional contact problem between a thermo-piezoelectric body and a thermally conductive foundation is numerically studied in this paper. The material's behaviour is described by means of a thermo-electro-elastic constitutive law. The process is quasistatic, the contact is bilateral and is associated to Tresca's law for dry friction. Hybrid formulation is introduced, it is a coupled system for the displacement field, the electric potential, the temperature and two Lagrange multipliers. The discrete scheme of the coupled system is introduced based on a finite element method to approximate the spatial variable and an Euler scheme to discretize the time derivate. The frictional contact is treated by using an augmented Lagrangian approach and a version of Newton's method. A solution algorithm is discussed and implemented. Finally, numerical simulation results are reported. These simulations show the performance of the algorithm and illustrate the effects of the conductivity of the foundation, as well.

Keywords: Augmented Lagrangian method; Bilateral contact; Finite element; Quasistatic process; Thermo-piezoelectric material.

1. INTRODUCTION

Piezoelectricity is a phenomenon which means that there is a coupling between the electrical and the mechanical state of the material. This coupling, in piezoelectric materials like crystals and ceramics, leads to the appearance of electric potential when a mechanical stress is applied and, conversely, mechanical strain is generated when electric potential is applied. The first effect is used in sensors, and the reverse effect is used in actuators, in engineering control equipments. Piezoelectric materials respond to temperature variations in a changing temperature environment. Thus, a coupling of thermo-electro-mechanical fields is needed to be taken into account if a temperature load is considered in a piezoelectric solid. These materials are used to detect the response of a structure by measuring the deformation induced due to applied electric potential or thermal variations. Thermal and piezoelectric effects have been included in the structural model and a fully coupled thermo-electro-elastic analysis has been performed, see [1-3]. Currently, there is a considerable mathematical interest in frictional contact problems involving thermo-piezoelectric materials, see for example [4-7] and the references therein. There, besides the rigorous construction of various mathematical models of contact for thermo-electro-elastic and thermo-electro-viscoelastic materials, the unique weak solvability of these models was proved, by using arguments of variational and hemivariational inequalities. However, no numerical simulation was studied. Recently, the numerical study of a model of static contact problem with Tresca's friction between a thermo-electro-elastic body and a rigid conductive foundation, considered in [8]. There, a discrete scheme to approximate the problem was considered and implemented in a numerical code, and numerical simulations were provided.

The present paper represents a continuation of [8] and it deals with a mathematical model which describes the frictional contact between a thermo-piezoelectric body and a conductive foundation. We use both the thermo-electro-elastic constitutive law and the friction conditions used in [8] but unlike [8], we assume here that the contact with the obstacle, the so-called foundation, is always produced and then, a bilateral contact condition is considered. This condition is other physical setting (see, e.g., [9, 10]). Also, note that, unlike [8]; here we consider a quasistatic process, which leads to an evolutionary model, different from the stationary model studied in [8]. Moreover, unlike [8], in the present paper we introduce a hybrid variational formulation of problem in which the dual variables corresponding to Lagrange multipliers are related to the contact stress and the friction force. The other trait of novelty of the present paper consists in the fact that here we deal with the numerical approach of the problem and provide numerical simulations. The corresponding numerical scheme is based on the spatial and temporal discretization. Furthermore, the spatial discretization is based on the finite element method, while the temporal discretization is based on the Euler scheme. Then, the scheme was utilized as a basis of a numerical code for the problem, in which we develop a specific contact element. We need this element in order to take into account the coupling of the mechanical and thermal unknows on the contact boundary condition. By using the code, simulation results on numerical example are presented.

The paper is organized as follows. In Section 2 the mechanical model is presented together with its variational formulation. A fully discrete scheme is presented in Section 3, and the numerical treatment of the frictional contact conditions is realized by using an augmented Lagrangian approach and a version of Newton's method. In Section 4, some numerical simulations are presented to highlight the performance of the method and the effects of the conductivity of the foundation, as well. Conclusions are finally drawn in section 5.

2. MECHANICAL PROBLEM AND VARIATIONAL FORMULATION

Consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d, d = 2, 3$ with a smooth boundary $\partial\Omega = \Gamma$. We use boldface letters for vectors and tensors, such as the outward unit normal on Γ, denoted by $\boldsymbol{\nu} = (\nu_i)$. The body is submitted to the action of body forces of density $\boldsymbol{f}_0$, a volume electric charges of density ϕ_0 and a volume heat source of constant strength q_0. It is also constrained mechanically, electrically and thermally on the boundary. To describe these constraints we consider a partition of Γ into three open disjoint parts Γ_D, Γ_N, Γ_C, on the one hand, and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas\Gamma_D > 0$ and $meas\Gamma_a > 0$. The body is clamped on Γ_D and therefore the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_N$ act on Γ_N. We also assume that the electrical potential vanishes on Γ_a and a surface electrical charge of density ϕ_b is prescribed on Γ_b. We assume that the temperature vanishes on $\Gamma_D \cup \Gamma_N$. In the reference configuration, the body is in contact over Γ_C with a conductive foundation. We assume that the foundation is thermally conductive and its temperature is maintained at θ_f. The contact is bilateral and is modelled with Tresca's law.

We use the notation S^d for the space of second order symmetric tensors on $\mathbb{R}^d$ and " **.** " and $\|.\|$ represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and S^d, respectively, that is $\boldsymbol{u}.\boldsymbol{v} = u_i v_i$, $||\boldsymbol{v}|| = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2}$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$ and $\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$, $||\boldsymbol{\tau}|| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2}$ for $\boldsymbol{\sigma}, \boldsymbol{\tau} \in S^d$. Here and everywhere in this paper i, j, k, l run from 1 to d, summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, i.e. $f_{,i} = \frac{\partial f}{\partial x_i}$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, by $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$, $\sigma_\nu = \sigma_{ij}\nu_i\nu_j$, and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$.
Thus, the classical model for the process is as in following.

Problem P. *Find a displacement* $\boldsymbol{u} : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \longrightarrow S^d$, *an electric potential* $\varphi : \Omega \times [0,T] \longrightarrow \mathbb{R}$, *an electric displacement field* $\boldsymbol{D} : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$, *a temperature field* $\theta : \Omega \times [0,T] \longrightarrow \mathbb{R}$ *and the heat flux* $\boldsymbol{q} : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ *such that*

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{F}}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\mathcal{E}}^T\boldsymbol{E}(\varphi) - \boldsymbol{\mathcal{M}}\theta \quad in\, \Omega \times (0,T), \tag{1}$$

$$\boldsymbol{D} = \boldsymbol{\mathcal{E}}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \boldsymbol{\beta}\boldsymbol{E}(\varphi) + \boldsymbol{\mathcal{P}}\theta \quad in\, \Omega \times (0,T), \tag{2}$$

$$\boldsymbol{q} = -\boldsymbol{\mathcal{K}}\nabla\theta \quad in\, \Omega \times (0,T), \tag{3}$$

$$Div\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad in\, \Omega \times (0,T), \tag{4}$$

$$div\, \boldsymbol{D} = \phi_0 \quad in\, \Omega \times (0,T), \tag{5}$$

$$\theta_{ref}\big(\alpha\dot{\theta} + \boldsymbol{\mathcal{M}}\varepsilon(\dot{\boldsymbol{u}}) + \boldsymbol{\mathcal{P}}\boldsymbol{E}(\dot{\varphi})\big) + div\, \boldsymbol{q} = q_0 \quad in\, \Omega \times (0,T), \tag{6}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad on\, \Gamma_D \times (0,T), \tag{7}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_N \quad on\, \Gamma_N \times (0,T), \tag{8}$$

$$u_\nu = 0, \quad ||\boldsymbol{\sigma}_\tau|| \le S, \quad \boldsymbol{\sigma}_\tau = -S\frac{\dot{\boldsymbol{u}}_\tau}{||\dot{\boldsymbol{u}}_\tau||} \quad if \quad \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{0} \quad on\, \Gamma_C \times (0,T), \tag{9}$$

$$\varphi = 0 \quad on\, \Gamma_a \times (0,T), \tag{10}$$

$$\boldsymbol{D}.\boldsymbol{\nu} = \phi_b \quad on\, \Gamma_b \times (0,T), \tag{11}$$

$$\theta = 0 \quad on\, \Gamma_D \cup \Gamma_N \times (0,T), \tag{12}$$

$$\boldsymbol{q}.\boldsymbol{\nu} = k_c(\theta - \theta_f) \quad on\, \Gamma_C \times (0,T), \tag{13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \varphi(0) = \varphi_0, \quad \theta(0) = \theta_0 \quad in\, \Omega. \tag{14}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x} \in \Omega \cup \Gamma$ and the time variable $t \in [0,T]$, where $T > 0$.

Equations (1) and (2) represent the thermo-electro-elastic constitutive law of the material in which denotes $\boldsymbol{\sigma} = (\sigma_{ij})$ the stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ denotes the linearized strain tensor, $\boldsymbol{E}(\varphi)$ is the electric field. $\boldsymbol{\mathcal{F}} = (f_{ijkl})$, $\boldsymbol{\mathcal{E}} = (e_{ijk})$, $\boldsymbol{\beta} = (\beta_{ij})$, $\boldsymbol{\mathcal{M}} = (m_{ij})$ and $\boldsymbol{\mathcal{P}} = (p_i)$ are respectively, the elasticity, piezoelectric, electric permittivity, thermal expansion and pyroelectric tensors. $\boldsymbol{\mathcal{E}}^T$ is the transpose of $\boldsymbol{\mathcal{E}}$. We recall that $\varepsilon_{ij}(\boldsymbol{u}) = (u_{i,j} + u_{j,i})/2$ and $\boldsymbol{E}(\varphi) = -\nabla\varphi = -(\varphi_{,i})$. Also the tensors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{E}}^T$ satisfy the equality $\boldsymbol{\mathcal{E}}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \boldsymbol{\mathcal{E}}^T\boldsymbol{v}$ $\forall\, \boldsymbol{\sigma} \in S^d,\ \boldsymbol{v} \in \mathbb{R}^d$, and the components of the tensor $\boldsymbol{\mathcal{E}}^T$ are given by $e_{ijk}^T = e_{kij}$. Notice, that Equation (1) takes into account the dependence of the stress on the electric potential and the temperature and Equation (2) describes a linear dependence of the electric displacement on the strain, the electric potential and the temperature. Equation (3) represents the Fourier law of heat conduction where $\boldsymbol{\mathcal{K}} = (k_{ij})$ denote the thermal conductivity tensor.

Equations (4)-(6) represent the equilibrium equations for the stress, the electric displacement and the heat flux fields, in which " Div " and " div " denote the divergence operators for tensor and vector valued functions, i.e. $Div\,\boldsymbol{\sigma} = (\sigma_{ij,i}),\ div\,\boldsymbol{Y} = (Y_{i,i})$. We use these equations since the process is assumed to be mechanically quasistatic, i.e., the inertia effects are neglected, electrically static, i.e., all radiation effects are neglected, and thermally quasistatic. In Equation (6) α is given as $\alpha = \rho c_v/\theta_{ref}$, where ρ is the mass density, c_v is the specific heat and θ_{ref} is the reference uniform temperature of the body. Equations (7)-(8), (10)-(11) and (12) represent the mechanical, electric and thermal boundary conditions. Equation (9) represents the bilateral contact with Tresca's friction law, where $S \geq 0$ is the friction bound, that is, the magnitude of the limiting friction traction at which slip begins. In Equation (9) the strong inequality holds in the stick zone and the equality in the slip zone. The contact is bilateral and that no separation takes place (see, e.g. [9, 10]). The temperature boundary condition in Equation (13) describes that the normal component of the heat flux is proportional to the difference between the temperature of the foundation θ_f and body's surface temperature where k_c is the coefficient of heat exchange between the body and the foundation, similar to that already used in [10]. Finally, the initial conditions $\boldsymbol{u}_0$, φ_0 and θ_0 in Equation (14) are given.

To present the variational formulation of Problem P we need some additional notation and preliminaries. We start by introducing the spaces $H = L^2(\Omega, \mathbb{R}^d)$, $\mathcal{H} = L^2(\Omega, S^d)$ and $H_1 = H^1(\Omega, \mathbb{R}^d)$. The spaces H, $\mathcal{H}$ and H_1 are Hilbert spaces equipped with the inner products $(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dx$, $(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx$ and $(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}$ respectively. The associated norms in H, $\mathcal{H}$ and H_1 are denoted by $||\cdot||_H$, $||\cdot||_{\mathcal{H}}$ and $||\cdot||_{H_1}$, respectively.

For the displacement, the electric potential and the temperature fields, we introduce the spaces $V = \{\boldsymbol{v} \in H^1(\Omega, \mathbb{R}^d);\ \boldsymbol{v} = \boldsymbol{0}\ \ on\ \Gamma_D\}$, $W = \{\psi \in H^1(\Omega);\ \psi = 0\ \ on\ \Gamma_a\}$ and $Q = \{\eta \in H^1(\Omega);\ \eta = 0\ \ on\ \Gamma_D \cup \Gamma_N\}$. On V, W and Q we consider the inner products and the corresponding norms given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = \big(\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\big)_{\mathcal{H}}, \quad ||\boldsymbol{v}||_V = ||\boldsymbol{\varepsilon}(\boldsymbol{v})||_{\mathcal{H}} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V, \tag{15}$$

$$(\varphi, \psi)_W = (\nabla\varphi, \nabla\psi)_H, \quad ||\psi||_W = ||\nabla\psi||_H \quad \text{for all } \varphi, \psi \in W, \tag{16}$$

$$(\theta, \eta)_Q = (\nabla\theta, \nabla\eta)_H, \quad ||\eta||_Q = ||\nabla\eta||_H \quad \text{for all } \theta, \eta \in Q. \tag{17}$$

Since $meas(\Gamma_D) > 0$ and $meas(\Gamma_a) > 0$, it is well known $(V, ||\cdot||_V)$, $(W, ||\cdot||_W)$ and $(Q, ||\cdot||_Q)$ are Hilbert spaces.

We consider the trace spaces $X_\nu = \{v_{\nu\backslash\Gamma_C} : \boldsymbol{v} \in V\}$ and $X_\tau = \{\boldsymbol{v}_{\tau\backslash\Gamma_C} : \boldsymbol{v} \in V\}$, equipped with their usual norms. Denote by X_ν^* and X_τ^* the duals of the spaces X_ν and X_τ, respectively. Moreover, we denote by $<\cdot,\cdot>_{X_\nu^* \times X_\nu}$ and $<\cdot,\cdot>_{X_\tau^* \times X_\tau}$ the corresponding duality pairing mappings. To establish the variational formulation, we need additional notations. Thus, we consider the four mappings $J : Q \times Q \longrightarrow \mathbb{R}$, $\boldsymbol{f} : [0, T] \longrightarrow V$, $\phi : [0, T] \longrightarrow W$ and $\Theta : [0, T] \longrightarrow Q$, defined by

$$J(\theta, \eta) = \int_{\Gamma_C} k_c(\theta - \theta_f)\eta\, da, \tag{18}$$

$$(\boldsymbol{f}(t), \boldsymbol{w})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{w}\, dx + \int_{\Gamma_N} \boldsymbol{f}_N(t) \cdot \boldsymbol{w}\, da, \tag{19}$$

$$(\phi(t), \psi)_W = \int_\Omega \phi_0(t)\psi\, dx - \int_{\Gamma_b} \phi_b(t)\psi\, da, \tag{20}$$

$$(\Theta(t), \eta)_Q = \int_\Omega q_0(t)\eta\, dx, \tag{21}$$

for all $\boldsymbol{w} \in V$, $\psi \in W$, $\eta \in Q$ and $t \in [0, T]$.

Then, the hybrid variational formulation of the contact problem P obtained by multiplying the equations with the relevant test functions and performing integration by part, is as follows.

Problem P_V. *Find a displacement* $\boldsymbol{u} : [0, T] \longrightarrow V$, *a normal stress* $\lambda_\nu : [0, T] \longrightarrow X_\nu^*$, *a tangential stress* $\boldsymbol{\lambda}_\tau : [0, T] \longrightarrow X_\tau^*$, *an electric potential* $\varphi : [0, T] \longrightarrow W$ *and a temperature* $\theta : [0, T] \longrightarrow Q$ *such that for a.e.* $t \in (0, T)$

$$\begin{aligned}(\boldsymbol{\mathcal{F}}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} &+ (\boldsymbol{\mathcal{E}}^T\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} - (\boldsymbol{\mathcal{M}}\theta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\ &+ <\lambda_\nu(t), v_\nu>_{X_\nu^* \times X_\nu} + <\boldsymbol{\lambda}_\tau(t), \boldsymbol{v}_\tau>_{X_\tau^* \times X_\tau} = (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V,\end{aligned} \tag{22}$$

$$(\boldsymbol{\beta}\nabla\varphi(t),\nabla\psi)_H - (\boldsymbol{\mathcal{E}}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\nabla\psi)_H - (\boldsymbol{\mathcal{P}}\theta(t),\nabla\psi)_H = (\phi(t),\psi)_W \quad \forall\psi \in W, \tag{23}$$

$$\left(\theta_{ref}\left(\alpha\dot{\theta}(t) + \boldsymbol{\mathcal{M}}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) - \boldsymbol{\mathcal{P}}\nabla\dot{\varphi}(t)\right),\eta\right)_{L^2(\Omega)} + (\boldsymbol{\mathcal{K}}\nabla\theta(t),\nabla\eta)_H + J(\theta(t),\eta) = (\Theta(t),\eta)_{L^2(\Omega)} \quad \forall\eta \in Q, \tag{24}$$

$$\lambda_\nu(t) \in \partial I_{\{0\}}\left(u_\nu(t)\right) \ in \ X_\nu^*, \tag{25}$$

$$\boldsymbol{\lambda}_\tau(t) \in \partial I^*_{C[S]}\left(\dot{\boldsymbol{u}}_\tau(t)\right) \ in \ X_\tau^*, \tag{26}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \ \varphi(0) = \varphi_0, \ \theta(0) = \theta_0. \tag{27}$$

The inclusion in Equation (25) represents the bilateral contact condition between the normal contact stress λ_ν and the normal gap distance u_ν. Here, $\partial I_{\{0\}}$ denotes subdifferential of the indicator function of the set $\{0\} \subset \mathbb{R}$. Recall also that, the inclusion in Equation (26) represents the subdifferential form of Tresca's law of dry friction. Here, $C[S]$ denotes the ball of radius S and ∂I_K^* denotes the subdifferential of the conjugate of the indicator function of the set K, see [11, 12] for details.

3. NUMERICAL APPROXIMATION AND SOLUTION ALGORITHM

3.1. Numerical approximation

We now present a fully discrete approximation of problem P_V. First, in order to approximate the special variable, we assume that Ω is a polygonal domain and we consider a regular triangulation of Ω, denoted by T^h, compatible with the boundary decomposition Γ_D, Γ_N and Γ_C. Here and below, $h > 0$ denotes the spatial discretization parameter. Consider the discrete variational spaces

$$V^h = \{\boldsymbol{v}^h \in [C(\overline{\Omega})]^d; \ \ \boldsymbol{v}^h{}_{\backslash T_{tr}} \in [P_1(T_{tr})]^d \ \ \forall T_{tr} \in T^h, \boldsymbol{v}^h = \boldsymbol{0} \ at \ the \ nodes \ of \ \Gamma_D\}, \tag{28}$$

$$W^h = \{\psi^h \in C(\overline{\Omega}); \ \ \psi^h{}_{\backslash T_{tr}} \in P_1(T_{tr}) \ \ \forall T_{tr} \in T^h, \psi^h = 0 \ at \ the \ nodes \ of \ \Gamma_a\}, \tag{29}$$

$$Q^h = \left\{\eta^h \in C(\overline{\Omega}); \ \ \eta^h{}_{\backslash T_{tr}} \in P_1(T_{tr}) \ \forall T_{tr} \in T^h, \eta^h = 0 \ at \ the \ nodes \ of \ \Gamma_D \cup \Gamma_N\right\}, \tag{30}$$

where $P_1(T_{tr})$ represents the space of polynomials of the global degree less or equal to one in T_{tr}. To discretize the time derivates, we use a uniform partition of $[0,T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N \leq T$ and let k be the time step size, $k = T/N$. In what follows, we denote $f_n = f(t_n)$. For a sequence $\{w_n\}_{n=0}^N$ we denote by $\delta w_n = (w_n - w_{n-1})/k$ the divided differences.

We now consider the spaces $X_\nu^h = \{ v^h_{\nu\backslash\Gamma_C} \ : \ v^h \in V^h\}$ and $X_\tau^h = \{ \boldsymbol{v}^h_{\tau\backslash\Gamma_C} \ : \ \boldsymbol{v}^h \in V^h\}$ equipped with their usual norm. We also consider the discrete space of piecewise constants $Y_\nu^h \subset L^2(\Gamma_C)$ and $Y_\tau^h \subset L^2(\Gamma_C)^d$ related to the discretization of the normal and the tangential stress, respectively.

The fully discrete approximation of problem P_V, based on the forward Euler scheme, is the following.

Problem P_V^{hk}. *Find a discrete displacement* $\boldsymbol{u}^{\boldsymbol{hk}} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$, *a discrete normal stress* $\lambda_\nu^{hk} = \left\{\lambda_{\nu_n}{}^{hk}\right\}_{n=0}^N \subset Y_\nu^h$, *a discrete tangential stress* $\boldsymbol{\lambda}_\tau^{hk} = \left\{\boldsymbol{\lambda}_{\tau_n}{}^{hk}\right\}_{n=0}^N \subset Y_\tau^h$, *a discrete electric potential* $\varphi^{hk} = \left\{\varphi_n^{hk}\right\}_{n=0}^N \subset W^h$ *and a discrete temperature* $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset Q^h$ *such that for all* $n = 1, \cdots, N$

$$(\boldsymbol{\mathcal{F}}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{v}^{\boldsymbol{h}}))_{\mathcal{H}} + (\boldsymbol{\mathcal{E}}^T\nabla\varphi_n^{hk},\boldsymbol{\varepsilon}(\boldsymbol{v}^{\boldsymbol{h}}))_{\mathcal{H}} - (\boldsymbol{\mathcal{M}}\theta_n^{hk},\boldsymbol{\varepsilon}(\boldsymbol{v}^{\boldsymbol{h}}))_{\mathcal{H}} + <\lambda_{\nu_n}{}^{hk}, v_\nu^h>_{X_\nu^*\times X_\nu} + <\boldsymbol{\lambda}_{\tau_n}{}^{hk},\boldsymbol{v}_\tau^h>_{X_\tau^*\times X_\tau} = \left(\boldsymbol{f}_n,\boldsymbol{v}^{\boldsymbol{h}}\right)_V \quad \forall\, \boldsymbol{v}^{\boldsymbol{h}} \in V^h, \tag{31}$$

$$(\boldsymbol{\beta}\nabla\varphi_n^{hk},\nabla\psi^h)_H - (\boldsymbol{\mathcal{E}}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}),\nabla\psi^h)_H - (\boldsymbol{\mathcal{P}}\theta_n^{hk},\nabla\psi^h)_H = (\phi_n,\psi^h)_W \quad \forall\psi^h \in W^h, \tag{32}$$

$$\left(\theta_{ref}(\alpha\delta\theta_n^{hk} + \boldsymbol{\mathcal{M}}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}_n^{hk}) - \boldsymbol{\mathcal{P}}\nabla\delta\varphi_n^{hk}),\eta^h\right)_{L^2(\Omega)} + (\boldsymbol{\mathcal{K}}\nabla\theta_n^{hk},\nabla\eta^h)_H + J(\theta_n^{hk},\eta^h) = (\Theta_n,\eta^h)_{L^2(\Omega)} \quad \forall\eta^h \in Q^h, \tag{33}$$

$$\lambda_{\nu_n}{}^{hk} \in \partial I_{\{0\}}\left(u_{\nu_n}{}^{hk}\right) \ in \ Y_\nu^h, \tag{34}$$

$$\boldsymbol{\lambda}_{\tau_n}{}^{hk} \in \partial I^*_{C[S]}\left(\delta\boldsymbol{u}_{\tau_n}{}^{hk}\right) \ in \ Y_\tau^h, \tag{35}$$

$$\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^{\boldsymbol{h}}, \ \varphi_0^{hk} = \varphi_0^h, \ \theta_0^{hk} = \theta_0^h. \tag{36}$$

Here $\boldsymbol{u}_0^h$, φ_0^h and θ_0^h are appropriate approximation of the initial condition $\boldsymbol{u}_0$, φ_0 and θ_0.

3.2. The solution algorithm

The frictional contact conditions in Equations (34) and (35) are treated by using a numerical approach based on the augmented Lagrangian approach, see [12, 13, 14]. To this end, we introduce the notation $\boldsymbol{\lambda} = \lambda_\nu \boldsymbol{\nu} + \boldsymbol{\lambda}_\tau$, where $\lambda_\nu = \boldsymbol{\lambda}\cdot\boldsymbol{\nu}$ and $\boldsymbol{\lambda}_\tau = \boldsymbol{\lambda} - \lambda_\nu \boldsymbol{\nu}$. Let N_{tot}^h be the total number of nodes and denote by α^i, β^i and γ^i the basis functions of the spaces V^h, W^h and Q^h, respectively, for $i = 1,\cdots,N_{tot}^h$. Then the expression of functions $\boldsymbol{v}^h \in V^h$, $\psi^h \in W^h$ and $\eta^h \in Q^h$ is given by $\boldsymbol{v}^h = \sum_{i=1}^{N_{tot}^h} \boldsymbol{v}^i\alpha^i$, $\psi^h = \sum_{i=1}^{N_{tot}^h}\psi^i\beta^i$ and $\eta^h = \sum_{i=1}^{N_{tot}^h}\eta^i\gamma^i$, where $\boldsymbol{v}^i$, ψ^i and η^i represent the values of the corresponding functions $\boldsymbol{v}^h$, ψ^h and η^h at the node of T^h.

The augmented Lagrangian approach shows as that the problem P_V^{hk} can be gouverned by the system of nonlinear equations

$$\boldsymbol{R}(\delta\boldsymbol{u}_n,\delta\varphi_n,\delta\theta_n,\boldsymbol{u}_n,\varphi_n,\theta_n,\boldsymbol{\lambda}_n) = \widetilde{\boldsymbol{A}}(\delta\boldsymbol{u}_n,\delta\varphi_n,\delta\theta_n) + \widetilde{\boldsymbol{G}}(\boldsymbol{u}_n,\varphi_n,\theta_n) + \widetilde{\boldsymbol{F}}(\boldsymbol{u}_n,\theta_n,\boldsymbol{\lambda}_n) = \boldsymbol{0}, \quad (37)$$

when the functions $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{G}}$ and $\widetilde{\boldsymbol{F}}$ are defined below.

Here, the vectors $\delta\boldsymbol{u}_n \in \mathbb{R}^{d\times N_{tot}^h}$, $\delta\varphi_n \in \mathbb{R}^{N_{tot}^h}$, $\delta\theta_n \in \mathbb{R}^{N_{tot}^h}$, $\boldsymbol{u}_n \in \mathbb{R}^{d\times N_{tot}^h}$, $\varphi_n \in \mathbb{R}^{N_{tot}^h}$, $\theta_n \in \mathbb{R}^{N_{tot}^h}$ and $\boldsymbol{\lambda}_n \in \mathbb{R}^{d\times N_{\Gamma_C}^h}$ are defined by $\delta\boldsymbol{u}_n = \{\delta\boldsymbol{u}_n^i\}_{i=1}^{N_{tot}^h}$, $\delta\varphi_n = \{\delta\varphi_n^i\}_{i=1}^{N_{tot}^h}$, $\delta\theta_n = \{\delta\theta_n^i\}_{i=1}^{N_{tot}^h}$, $\boldsymbol{u}_n = \{\boldsymbol{u}_n^i\}_{i=1}^{N_{tot}^h}$, $\varphi_n = \{\varphi_n^i\}_{i=1}^{N_{tot}^h}$, $\theta_n = \{\theta_n^i\}_{i=1}^{N_{tot}^h}$ and $\boldsymbol{\lambda}_n = \{\boldsymbol{\lambda}_n^i\}_{i=1}^{N_{\gamma_C}^h}$, where $\delta\boldsymbol{u}_n^i := \frac{\boldsymbol{u}_n^i - \boldsymbol{u}_{n-1}^i}{k}$, $\delta\varphi_n^i := \frac{\varphi_n^i - \varphi_{n-1}^i}{k}$, $\delta\theta_n^i := \frac{\theta_n^i - \theta_{n-1}^i}{k}$, $\boldsymbol{u}_n^i$, φ_n^i and θ_n^i denote the values of functions $\delta\boldsymbol{u}_n^{hk}$, $\delta\varphi_n^{hk}$, $\delta\theta_n^{hk}$, $\boldsymbol{u}_n^{hk}$, φ_n^{hk} and θ_n^{hk} at the i^{th} node of T^h. Moreover, $\boldsymbol{\lambda}_n^i$ represents the value of $\boldsymbol{\lambda}_n^{hk}$ at the i^{th} node of the discretized contact interface, where $N_{\Gamma_C}^h$ denotes the total number of nodes of T^h lying on Γ_C.

Next, the generalized damping term $\widetilde{\boldsymbol{A}}(\delta\boldsymbol{u}_n,\delta\varphi_n,\delta\theta_n) \in \mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{d\times N_{\Gamma_C}^h}$ and the genaralized themo-electro-elastic term $\widetilde{\boldsymbol{G}}(\boldsymbol{u}_n,\varphi_n,\theta_n) \in \mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{d\times N_{\Gamma_C}^h}$ are defined by $\widetilde{\boldsymbol{A}}(\delta\boldsymbol{u}_n,\delta\varphi_n,\delta\theta_n) = (\mathbf{A}(\delta\boldsymbol{u}_n,\delta\varphi_n,\delta\theta_n),\boldsymbol{0}_{d\times N_{\Gamma_C}^h})$ and $\widetilde{\boldsymbol{G}}(\boldsymbol{u}_n,\varphi_n,\theta_n) = (\boldsymbol{G}(\boldsymbol{u}_n,\varphi_n,\theta_n),\boldsymbol{0}_{d\times N_{\Gamma_C}^h})$. Here $\boldsymbol{0}_{d\times N_{\Gamma_C}^h}$ is the zero element of $\mathbb{R}^{d\times N_{\Gamma_C}^h}$; also, $\mathbf{A}(\delta\boldsymbol{u}_n,\delta\varphi_n,\delta\theta_n) \in \mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}$ and $\boldsymbol{G}(\boldsymbol{u}_n,\varphi_n,\theta_n) \in \mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}$ denote the damping term and the thermo-electro-elastic term, respectively, given by

$$\left(\boldsymbol{A}(\delta\boldsymbol{u},\delta\varphi,\delta\theta)\cdot(\boldsymbol{v},\psi,\eta)\right)_{\mathbb{R}^{d\times N_{tot}}\times\mathbb{R}^{N_{tot}}\times\mathbb{R}^{N_{tot}}} = \left(\theta_{ref}(\alpha\delta\theta^h + \boldsymbol{\mathcal{M}}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}^h) - \boldsymbol{\mathcal{P}}\nabla\delta\varphi^h),\eta^h\right)_{L^2(\Omega)}, \quad (38)$$

$$\begin{aligned}
&\left(\boldsymbol{G}(\boldsymbol{u},\varphi,\theta)\cdot(\boldsymbol{v},\psi,\eta)\right)_{\mathbb{R}^{d\times N_{tot}}\times\mathbb{R}^{N_{tot}}\times\mathbb{R}^{N_{tot}}} \\
&= (\boldsymbol{\mathcal{F}}\boldsymbol{\varepsilon}(\boldsymbol{u}^h),\boldsymbol{\varepsilon}(\boldsymbol{v}^h))_{\mathcal{H}} + (\boldsymbol{\mathcal{E}}^T\nabla\varphi^h,\boldsymbol{\varepsilon}(\boldsymbol{v}^h))_{\mathcal{H}} - (\boldsymbol{\mathcal{M}}\theta^h,\boldsymbol{\varepsilon}(\boldsymbol{v}^h))_{\mathcal{H}} - \left(\boldsymbol{f}_n,\boldsymbol{w}^h\right)_V \quad (39)\\
&- (\boldsymbol{\mathcal{E}}\boldsymbol{\varepsilon}(\boldsymbol{u}^h) - \boldsymbol{\beta}\nabla\varphi^h,\nabla\psi^h)_H - (\boldsymbol{\mathcal{P}}\theta^h,\nabla\psi^h)_H - \left(\phi_n,\psi^h\right)_W + (\boldsymbol{\mathcal{K}}\nabla\theta^h,\nabla\eta^h)_H \\
&- (\Theta_n,\eta^h)_{L^2(\Omega)},
\end{aligned}$$

$\forall\, \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^{d\times N_{tot}}, \forall\, \varphi,\psi \in \mathbb{R}^{N_{tot}}, \forall\, \theta,\eta \in \mathbb{R}^{N_{tot}}, \forall\, \boldsymbol{u}^h,\boldsymbol{v}^h \in V^h, \forall\, \varphi^h,\psi^h \in W^h, \forall\, \theta^h,\eta^h \in Q^h.$

Above, $\boldsymbol{v}$, ψ and η represents the generalized vector of components $\boldsymbol{v}^i$, ψ^i and η^i for $i = 1,\cdots,N_{tot}^h$, respectivelly. Finally, the generalized contact operator $\widetilde{\boldsymbol{F}}(\boldsymbol{u}_n,\theta_n,\boldsymbol{\lambda}_n) \in \mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{d\times N_{\Gamma_C}^h}$ is defined by $\widetilde{\boldsymbol{F}}(\boldsymbol{u}_n,\theta_n,\boldsymbol{\lambda}_n) = (\nabla_{\boldsymbol{u}}(l_\nu^r + l_\tau^r), 0_{N_{tot}^h}, k_c(\theta - \theta_f), \nabla_\lambda(l_\nu^r + l_\tau^r))$, where ∇_x represents the gradient operator with respect the variable x, and $0_{N_{tot}^h}$ is the zero element of $\mathbb{R}^{N_{tot}^h}$. Also, l_ν^r and l_τ^r denote the augmented Lagrangian functionals

$$l_\nu^r(\boldsymbol{u}^h,\lambda_\nu^h) = u_\nu^h\lambda_\nu^h + \frac{r_\nu}{2}(u_\nu^h)^2, \quad (40)$$

$$l_\tau^r(\boldsymbol{u}^h,\boldsymbol{\lambda}_\tau^h) = \delta\boldsymbol{u}_\tau^h\cdot\boldsymbol{\lambda}_\tau^h + \frac{r_\tau}{2}||\delta\boldsymbol{u}_\tau^h||^2 - \frac{1}{2r_\tau}\, dist^2\{\boldsymbol{\lambda}_\tau^h + r_\tau\delta\boldsymbol{u}_\tau^h,\ C[S]\}, \quad (41)$$

where r_ν and r_τ are positive penalty coefficients and $dist\{x,C\}$ denotes the distance from x to the set C, i.e., $dist(x,C) = \inf_{y\in C}||x-y||$. Also, θ_f^h represents an approximation of the temperature of the foundation.

Let $\boldsymbol{F}(\boldsymbol{u}_n,\theta_n,\boldsymbol{\lambda}_n) \in \mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{d\times N_{\Gamma_C}^h}$ the contact operator defined through the relation

$$\begin{aligned}
&(\boldsymbol{F}(\boldsymbol{u}_n,\theta_n,\boldsymbol{\lambda}_n),\boldsymbol{v},\eta,\boldsymbol{\xi})_{\mathbb{R}^{d\times N_{tot}^h}\times\mathbb{R}^{N_{tot}^h}\times\mathbb{R}^{d\times N_{\Gamma_C}^h}} \\
&= \int_{\Gamma_C}\nabla_{\boldsymbol{u}}(l_\nu^r + l_\tau^r)\cdot\boldsymbol{v}^h\, da + \int_{\Gamma_C}\nabla_\lambda(l_\nu^r + l_\tau^r)\cdot\boldsymbol{\gamma}^h\, da + \int_{\Gamma_C}k_c(\theta^h - \theta_f^h)\eta^h\, da, \quad (42)
\end{aligned}$$

$\forall\, \boldsymbol{v} \in \mathbb{R}^{d\times N_{tot}},\ \forall\, \eta \in \mathbb{R}^{N_{tot}},\ \forall\, \boldsymbol{\xi} \in \mathbb{R}^{d\times N_{\Gamma_C}^h},\ \forall\, \boldsymbol{v}^h \in V^h,\ \forall\, \theta^h,\eta^h \in Q^h,\ \forall\, \boldsymbol{\gamma}^h \in Y_\nu^h\times Y_\tau^h.$

The solution algorithm consists in a combination between the finite differences (backward Euler difference) and the linear iterations methods (Newton method). The finite difference scheme we use is characterized by a first order time integration scheme, both for the velocity $\delta\boldsymbol{u}_n$ and $\delta\theta_n$. To solve Equation (37), at each time increment the variables $(\boldsymbol{u}_n, \varphi_n, \theta_n, \boldsymbol{\lambda}_n)$ are treated simultaneously through a Newton method and, for this reason, we use in what follows the notation $\boldsymbol{x}_n = (\boldsymbol{u}_n, \varphi_n, \theta_n, \boldsymbol{\lambda}_n)$.

Inside the loop of the increment time indexed by n, the algorithm we use can be developed in three steps which are the following.

- **For** $n = 0$ **until** N, let $\boldsymbol{u}_0$, φ_0, θ_0 and $\boldsymbol{\lambda}_0$ be given.
- **A prediction step**: This step provides the initial values $\boldsymbol{u}^0_{n+1}$, φ^0_{n+1}, θ^0_{n+1}, δu^0_{n+1}, $\delta\varphi^0_{n+1}$, $\delta\theta^0_{n+1}$ and $\boldsymbol{\lambda}^0_{n+1}$ by the formulas: $\boldsymbol{u}^0_{n+1} = \boldsymbol{u}_n$, $\varphi^0_{n+1} = \varphi_n$, $\theta^0_{n+1} = \theta_n$, $\boldsymbol{\lambda}^0_{n+1} = \boldsymbol{\lambda}_n$, $\boldsymbol{\delta u}^0_{n+1} = \boldsymbol{0}$, $\delta\varphi^0_{n+1} = \varphi_n$ and $\delta\theta^0_{n+1} = \theta_n$.
- **A Newton linearization step**: for $i = 0$ until convergence, compute

$$\boldsymbol{x}^{\mathrm{i+1}}_{\mathrm{n+1}} = \boldsymbol{x}^{\mathrm{i}}_{\mathrm{n+1}} - \left(\frac{\boldsymbol{Q}^i_{n+1}}{k} + \boldsymbol{K}^i_{n+1} + \boldsymbol{T}^i_{n+1}\right)^{-1} \boldsymbol{R}\left(\frac{\boldsymbol{u}^i_{n+1} - \boldsymbol{u}^i_n}{k}, \frac{\varphi^i_{n+1} - \varphi^i_n}{k}, \frac{\theta^i_{n+1} - \theta^i_n}{k}, \boldsymbol{u}_n, \varphi_n, \theta_n, \boldsymbol{\lambda}_n\right), \quad (43)$$

where $\boldsymbol{x}^{i+1}_{n+1}$ denotes the pair $(\boldsymbol{u}^{i+1}_{n+1}, \varphi^{i+1}_{n+1}, \theta^{i+1}_{n+1}, \boldsymbol{\lambda}^{i+1}_{n+1})$; i and n represent respectively the Newton iteration index and the time index; $\boldsymbol{Q}^i_{n+1} = D_{\boldsymbol{u},\varphi,\theta}\boldsymbol{A}(\delta\boldsymbol{u}^i_{n+1}, \delta\varphi^i_{n+1}, \delta\theta^i_{n+1})$ denotes the damping matrix, $\boldsymbol{K}^i_{n+1} = D_{\boldsymbol{u},\varphi,\theta}\boldsymbol{G}(\boldsymbol{u}^i_{n+1}, \varphi^i_{n+1}, \theta^{i+1}_{n+1})$ represents the thermo-electro-elastic matrix and $\boldsymbol{T}^i_{n+1} = D_{\boldsymbol{u},\theta,\boldsymbol{\lambda}}\boldsymbol{F}(\boldsymbol{u}^i_{n+1}, \theta^i_{n+1}, \boldsymbol{\lambda}^i_{n+1})$ is the contact tangent matrix; also, $D_{\boldsymbol{u},\varphi,\theta}\boldsymbol{A}$, $D_{\boldsymbol{u},\varphi,\theta}\boldsymbol{G}$ and $D_{\boldsymbol{u},\theta,\boldsymbol{\lambda}}\boldsymbol{F}$ denote the differentials of the functions $\boldsymbol{A}, \boldsymbol{G}$ and $\boldsymbol{F}$ with respect to the variables $\boldsymbol{u}$, φ, θ and $\boldsymbol{\lambda}$. This leads us to solve the resulting linear system

$$\left(\frac{\boldsymbol{Q}^i_{n+1}}{k} + \boldsymbol{K}^i_{n+1} + \boldsymbol{T}^i_{n+1}\right)\Delta\boldsymbol{x}^i = -\boldsymbol{R}\left(\frac{\boldsymbol{u}^i_{n+1} - \boldsymbol{u}^i_n}{k}, \frac{\varphi^i_{n+1} - \varphi^i_n}{k}, \frac{\theta^i_{n+1} - \theta^i_n}{k}, \boldsymbol{u}_n, \varphi_n, \theta_n, \boldsymbol{\lambda}_n\right), \quad (44)$$

where $\Delta\boldsymbol{x}^i = (\Delta\boldsymbol{u}^i, \Delta\varphi^i, \Delta\theta^i, \Delta\boldsymbol{\lambda}^i)$ with $\Delta\boldsymbol{u}^i = \boldsymbol{u}^{i+1}_{n+1} - \boldsymbol{u}^i_{n+1}$, $\Delta\varphi^i = \varphi^{i+1}_{n+1} - \varphi^i_{n+1}$, $\Delta\theta^i = \theta^{i+1}_{n+1} - \theta^i_{n+1}$ and $\Delta\boldsymbol{\lambda}^i = \boldsymbol{\lambda}^{i+1}_{n+1} - \boldsymbol{\lambda}^i_{n+1}$.

- **A correction step**: Once the system in Equation (44) is resolved, we update $\boldsymbol{x}^{i+1}_{n+1}$, $\boldsymbol{u}^{i+1}_{n+1}$, φ^{i+1}_{n+1}, θ^{i+1}_{n+1} and $\boldsymbol{\lambda}^{i+1}_{n+1}$ by $\boldsymbol{x}^{i+1}_{n+1} = \boldsymbol{x}^i_{n+1} + \Delta\boldsymbol{x}^i$, $\boldsymbol{u}^{i+1}_{n+1} = \boldsymbol{u}^i_{n+1} + \Delta\boldsymbol{u}^i$, $\varphi^{i+1}_{n+1} = \varphi^i_{n+1} + \Delta\varphi^i$, $\theta^{i+1}_{n+1} = \theta^i_{n+1} + \Delta\theta^i$ and $\boldsymbol{\lambda}^{i+1}_{n+1} = \boldsymbol{\lambda}^i_{n+1} + \Delta\boldsymbol{\lambda}^i$.

Note that the formulation in Equation (37) has been implemented in the open-source finite element library GetFEM++ (see [15]).

4. NUMERICAL SIMULATIONS

Now we illustrate our theoretical results by numerical simulations in the study of two-dimensional test problem. In order to observe the effect of the thermo-piezoelectric properties of the material, a physical setting as the one depicted in Figure 1 is considered. Where, $\Omega = (0, 0.1) \times (0, 0.1)$ and $\Gamma_D = \Gamma_a = \{0.1\} \times [0, 0.1]$, $\Gamma_N = \Gamma_b = ([0, 0.1] \times \{0\}) \cup ([0, 0.1] \times \{0.1\})$, $\Gamma_C = \{0\} \times [0, 0.1]$. On Γ_D the body is clamped and the electric potential is free there. The body is subjected to the action of surface tractions acting on $[0, 0.1] \times \{0.1\}$, i.e., $\boldsymbol{f}_N(x_1, x_2, t) = (0, -0.5\, t) N/m$, while the remainder of the part Γ_N is free. We assume that the temperature vanishes on $\Gamma_D \cup \Gamma_N$. The body is in bilateral frictional contact with a conductive foundation on the part Γ_C of the boundary. The following data have been used in the numerical simulations:

$$r_\nu = r_\tau = 10^9\, N/m^2, \quad S = 15\, N/m^2, \quad k_c = 1\ W/m^2K, \quad \theta_f = -27\, K, \quad \theta_{ref} = 293\, K.$$
$$\boldsymbol{f}_0 = \boldsymbol{0}\, N/m^2, \quad \phi_0 = 0\ C/m^2, \quad \phi_b = 0\ C/m, \quad q_0 = 0\, W/m^2, \quad q_b = 0\, W/m.$$
$$T = 0.2\, s, \quad \boldsymbol{u}_0 = \boldsymbol{0}\, m, \quad \varphi_0 = 0\, V, \quad \theta_0 = 0\, K.$$

In the plane of deformations setting, the constitutive law in Equations (1) and (2) can be written by using a compressed matrix notation in place of the tensor notation as follows

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{33}\\ \sigma_{13}\\ D_1\\ D_3\end{bmatrix} = \begin{bmatrix} f_{11} & f_{13} & 0 & 0 & e_{31}\\ f_{13} & f_{33} & 0 & 0 & e_{33}\\ 0 & 0 & f_{44} & e_{15} & 0\\ 0 & 0 & e_{15} & -\beta_{11} & 0\\ e_{31} & e_{33} & 0 & 0 & -\beta_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{33}\\ 2\varepsilon_{13}\\ -E_1\\ -E_3\end{bmatrix} + \begin{bmatrix}-m_{11}\\ -m_{33}\\ 0\\ -p_1\\ -p_3\end{bmatrix}\theta.$$

The material parameters of B_aTiO_3 are taken as (see [2]):

- Elastic $[GPa]$: $f_{11} = 166$, $f_{13} = 78$, $f_{33} = 162$, $f_{44} = 43$;
- Piezoelectric $[C/m^2]$: $e_{31} = -4.4$, $e_{33} = 18.6$, $e_{15} = 11.6$;
- Dielectric $[C/GVm]$: $\beta_{11} = 11.2$, $\beta_{33} = 12.6$;
- Thermal expansion $[\times 10^6\ \mathrm{N/Km^2}]$: $m_{11} = 2.24$, $m_{33} = 1.89$;

- Pyroelectric [$\times 10^{-4}$ C/Km^2]: $p_1 = 0,\ p_3 = -1$;
- Heat conduction coefficients [W/Km]: $k_{11} = 50,\ k_{33} = 75$;
- $\rho = 5500$ [kg/m^3]; $c_v = 420$ [Ws/kgK].

Our interest in this example is to study the influence of the thermal conductivity of the foundation on the contact process and, to this end, we consider the problem both in the case when the foundation is thermally insulated and, in the case, when it is thermally conductive. When the foundation is insulated there are no heat flux on Γ_C (i.e. $\boldsymbol{q} \cdot \boldsymbol{\nu} = 0$ on Γ_C) and when the foundation is conductive: the normal component of the heat flux is proportional to the difference between the temperature of the foundation and body's surface temperature (i.e. $\boldsymbol{q}.\boldsymbol{\nu} = k_c\big(\theta - \theta_f\big)$ on Γ_C). Our results are shown in Figures 2-4.

Figure 2 presents the deformed configurations for the two previously mentioned cases, at final time T. Note that in the case of an insulated foundation, the body is compressed by the actions of tractions. However, in the case of a conductive foundation, the shape of the body changes greatly because of the difference in the temperature. In order to highlight the influence of the foundation temperature on the electric potential, we plot the electric potential for the two previously mentioned case (see figure 3). The first case illustrates the direct piezoelectric effect: the electric potential is generated because of the deformation and its higher values are located on the top vertical extremities of Γ_C. However, in the second case, we can easily note that considering a thermally conductive foundation increases the electric potential and its higher values are located on the bottom vertical extremities of Γ_C.

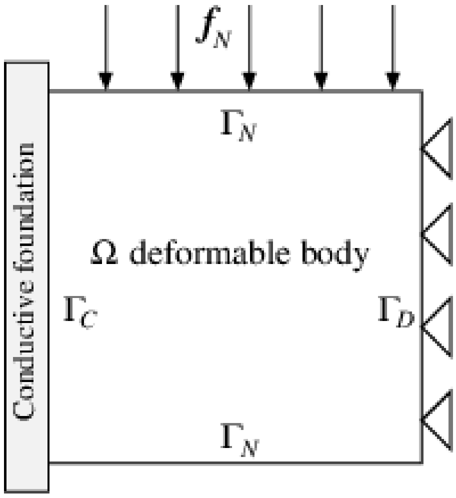


Figure 1. Physical setting

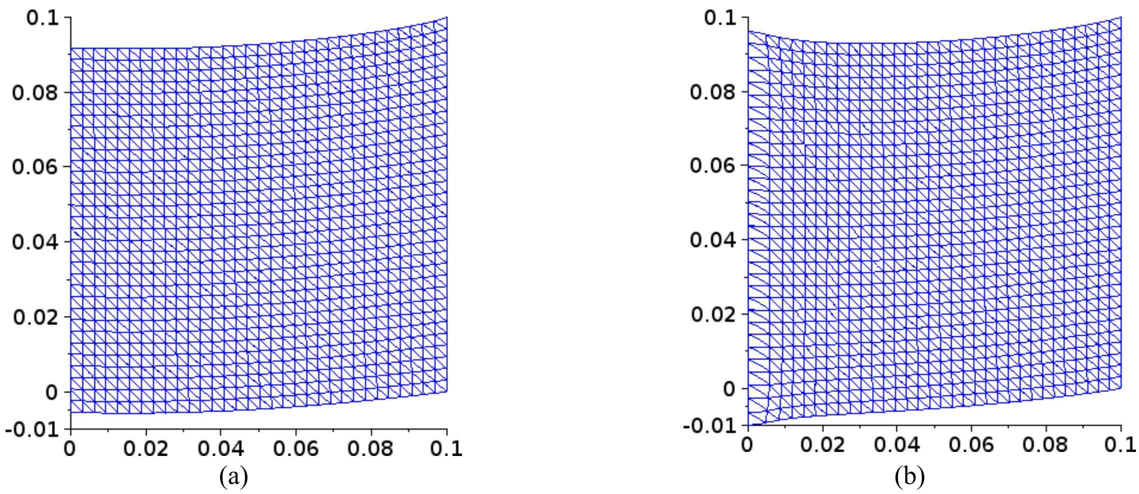
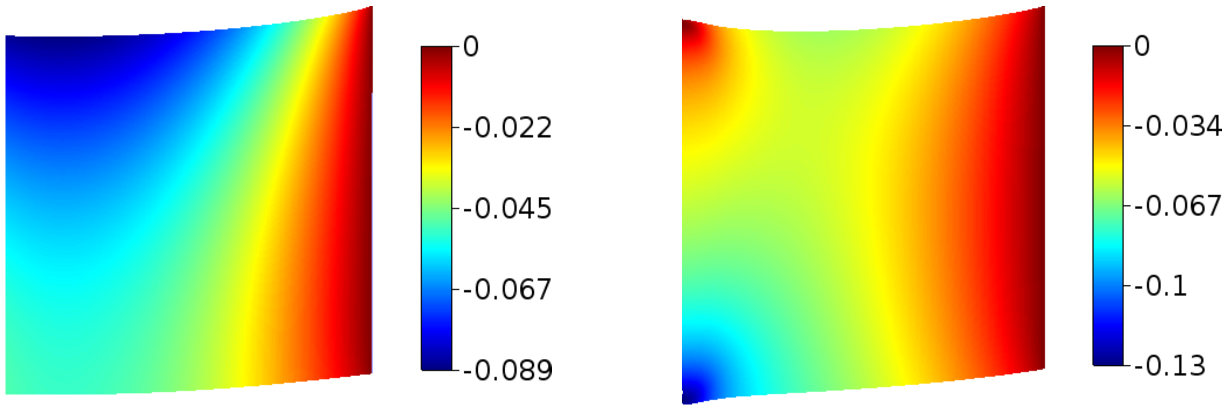
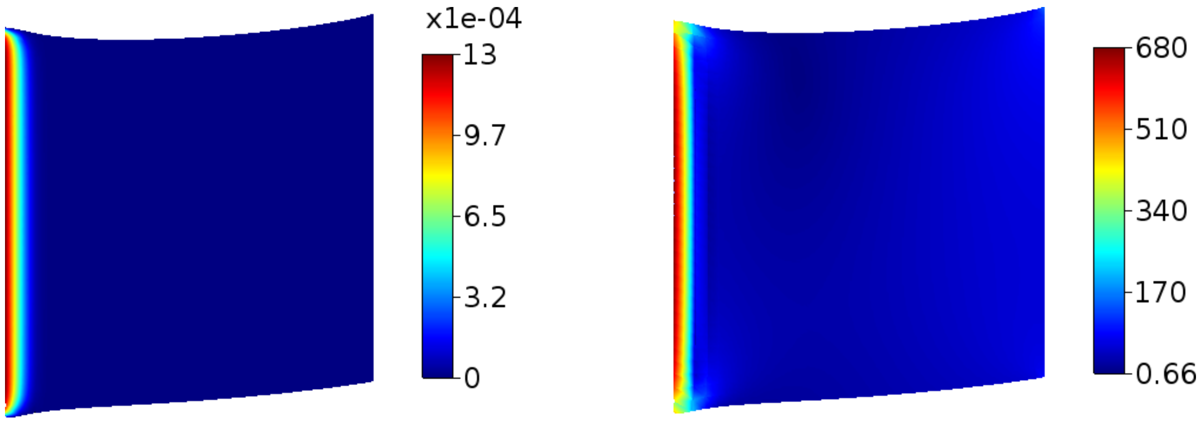


Figure 2. Amplified deformed mesh in the case of an insulated foundation (a) and in the case of a conductive foundation (b)



(a) (b)

Figure 3. Electric potential [V] in the case of (a) an insulated foundation (b) a conductive foundation



(a) (b)

Figure 4. (a) Temperature field [K] and (b) the Von Mises stress norm [Pa] at final time

The values of the temperature and the Von Mises norm of the stress field in the deformed configuration are plotted in Figure 4 for the value $\theta_f = -27\ K$. One can observe that heat is generated at the contact interface due to the assumption that the foundation is thermally conductive, and is diffusing into the body (see Figure 4 (a)). Moreover, we note that the higher values of the elastic stresses are localised on the zones of the contact (Figure 4 (b)).

5. CONCLUSION

This paper provides a numerical study of a newly formulated model in contact mechanics. The model concerns frictional contact for thermo-piezoelectric materials. The novelties arise in the fact that the process is quasistatic, the material behavior is described by a thermo-electro-elastic constitutive law and the foundation is thermally conductive. A fully discrete scheme was used to approach the problem and a numerical algorithm which combine the augmented Lagrangian approach with the Newton method was implemented. Moreover, numerical simulations for a representative two-dimensional example were provided in the two different cases insulating and conducting thermal boundary conditions. These simulations describe the thermal effect, i.e. the appearance of strain and voltage in the body, due to the action of the temperature flied. Also, they underline the effects of the thermal conductivity of the foundation on the process. Performing these simulations, we found that the numerical solution worked well and the convergence was rapid. This work opens the way to study further problems with other conditions for thermally-electrically conductive foundation, in a dynamic case.

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